W13 - Notes

Parametric curves

Videos

Videos, Organic Chemistry Tutor

- [Intro to parametric equations and graphing]()

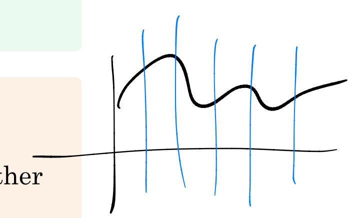
01 Theory

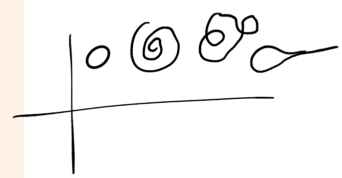
Parametric curves are curves traced by the path of a 'moving' point. An independent parameter, such as t for 'time', controls *both x and y* values through **Cartesian coordinate functions** $x(t)$ and $y(t)$. The coordinates of the moving point are $(x(t), y(t))$.

graph of $f(x)$:

$\{ (x, f(x)) \}$

$\{ (x,y) \mid y = f(x) \}$





Parametric curve

A **parametric curve** is a function from parameter space $\mathbb{R}$ to the plane $\mathbb{R}^2$ given in terms of coordinate functions:

$$t \longmapsto \big(x(t),\, y(t) \big)$$

Other notations

Be aware that sometimes the coordinate functions are written with f and g (or yet other letters) like this:

$$(x, y) = \big(f(t),\, g(t) \big)$$

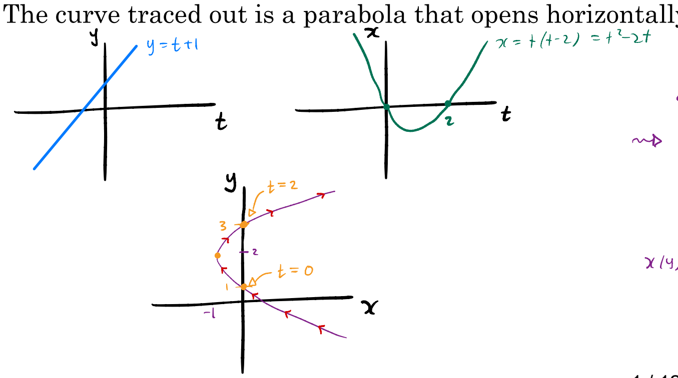
Or simply equating coordinate letters with functions: $x = f(t), \quad y = g(t)$

Sometimes a different parameter is used, like s or u.

For example, suppose:

$$x = \underbrace{t^2 - 2t}_{=t(t-2)}, \qquad y = t + 1$$

The curve traced out is a parabola that opens horizontally:



Solve: $t = y - 1$

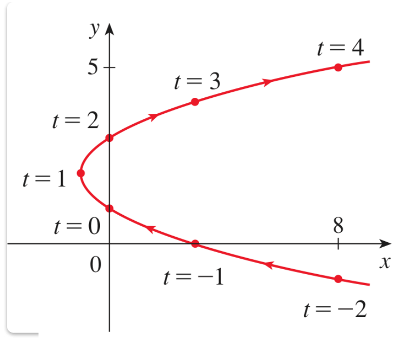
$\rightsquigarrow x = (y-1)^2 - 2(y-1)$

$= y^2 - 2y + 1 - 2y + 2$

$= y^2 - 4y + 3$

$= (y^2 - 4y + 4) - 1$

$x(y) = (y-2)^2 - 1$



Given a parametric curve, we can create an equation satisfied by x and y variables by solving for t in either coordinate function (inverting either f or g) and plugging the result into the other function.

In the example:

$$y = t + 1 \quad \ggg \quad t = y - 1$$

$$\ggg \quad x = t^2 - 2t \quad \ggg \quad x = (y-1)^2 - 2(y-1)$$

$$\ggg \quad x = y^2 - 4y + 3 \quad \ggg \quad x = (y-2)^2 - 1$$

This is the equation of a parabola centered at $(-1, 2)$ that opens to the right.

Image of a parametric curve

The **image** of a parametric curve is the *set* of output points $\big(x(t),\, y(t)\,\big)$ that are traversed by the moving point.

A parametric curve has *hidden information* that isn't contained in the image:

- The *time values* t when the moving point is found in various locations.
- The *speed* at which the curve is traversed.
- The *direction* in which the curve is traversed.

We can **reparametrize** a parametric curve to use a different parameter or different coordinate functions while leaving the *image unchanged*.

$t \rightsquigarrow t+1$

In the previous example, shift t by 1:

$$x = (t+1)^2 - 2(t+1), \qquad y = (t+1) + 1$$

$$\ggg \qquad x = t^2 - 1, \qquad y = t + 2$$

Since the parameter t and the parameter $t+1$ both cover the same values for $t \in (-\infty, \infty)$, the same curve is traversed. But the moving point in the second, shifted version reaches any given location *one unit earlier* in time. (When $t = -1$ in the second version, the input to $x(t)$ and $y(t)$ is the same as when $t = 0$ in the first one.)

02 Illustration

From: $\cos^2 t + \sin^2 t = 1$

$(R\cos t)^2 + (R\sin t)^2 = R^2$

$(R\cos t, R\sin t) = (x, y)$

Ellipse:

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$x = a\cos t$
$y = b\sin t$

☰ Example - Parametric circles

The standard equation of a circle of radius R centered at the point (h, k):

$$(x-h)^2 + (y-k)^2 = R^2$$

This equation says that the *distance* from a point (x, y) on the circle to the center point (h, k) equals R. This fact defines the circle.

Parametric coordinates for the circle:

$$x = h + R\cos t, \qquad y = k + R\sin t, \qquad t \in [0, 2\pi)$$

For example, the unit circle $x^2 + y^2 = 1$ is parametrized by $x = \cos t$ and $y = \sin t$.

☰ Example - Parametric lines

(1) Parametric coordinate functions for a line:

$(a, b) = (x, y)$ @ $t = 0$

$$x = a + rt, \qquad y = b + st, \qquad t \in (-\infty, +\infty)$$

Compare this to the graph of linear function:

$$y = \underset{= \frac{s}{r}}{m}x + b \qquad \text{some } m, b$$

Vertical lines cannot be described as the graph of a function. We must use $x = a$.

vertical line: "$x = a$" from: $a = a$, $r = 0$ $\quad$ $b =$ anything, $s \neq 0$ $\quad$ $x = a$, $y = b + st$

---

(2) Parametric lines can describe all lines equally well, including horizontal and vertical lines.

A vertical line $x = a$ is achieved by setting $s = 0$ and $r \neq 0$.

A horizontal line $y = b$ is achieved by setting $r = 0$ and $s \neq 0$.

A non-vertical line $y = mx + b$ may be achieved by setting $s = m$ and $r = 1$, and $a = 0$.

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(3) Assuming that $r \neq 0$, the parametric coordinate functions describe a line satisfying:

$x = a + rt$
$x - a = rt$
$\frac{x-a}{r} = t$

$y = b + st$
$= b + s\left(\frac{x-a}{r}\right)$

$$y = b + s\left(\frac{x-a}{r}\right)$$

$$\gg\gg \quad y = \frac{s}{r} \cdot x + \left(b - \frac{s}{r} \cdot a\right)$$

and therefore the slope is $m = \frac{s}{r}$ and the y-intercept is $b - \frac{s}{r} \cdot a$.

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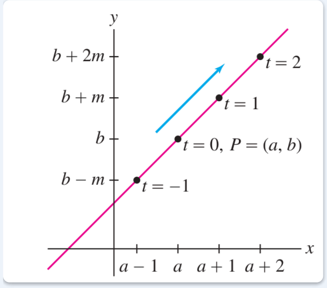
(4) The point-slope construction of a line has a parametric analogue:

point-slope line:

$$y - a = m(x - b) \qquad\qquad (x, y) = (a + t,\, b + mt)$$



Example - Parametric ellipses

The general equation of an ellipse centered at (h, k) with half-axes a and b is:

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

This equation represents a *stretched unit circle*:

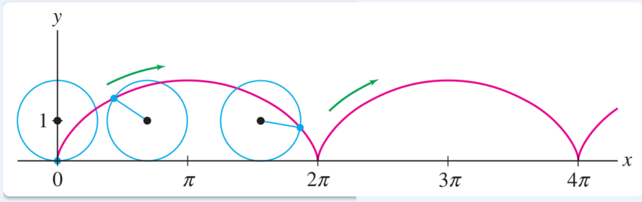
- by a in the x-axis
- by b in the y-axis

Parametric coordinate functions for the general ellipse:

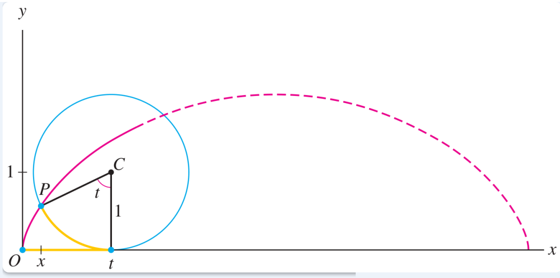
$$x = h + a\cos t, \qquad y = k + b\sin t, \qquad t \in [0, 2\pi)$$

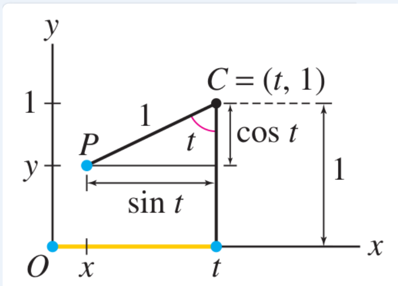
Example - Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:





The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t, 1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t - \sin t$. The y coordinate is $1 - \cos t$.

So the coordinates of the point $P = (x, y)$ are given parametrically by:

$$x = t - \sin t, \qquad y = 1 - \cos t, \qquad t > 0$$

If the circle has another radius, say R, then the parametric formulas change to:

$$x = Rt - R\sin t, \qquad y = R - R\cos t, \qquad t > 0$$

Calculus with parametric curves

03 Theory - Slope, concavity

We can use $x(t)$ and $y(t)$ data to compute the slope of a parametric curve in terms of t.

Slope formula

Given a parametric curve $\big(x(t),\, y(t)\big)$, its slope satisfies:

$$\frac{y'}{x'} = \frac{dy/dt}{dx/dt} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \qquad (\text{where } x'(t) \neq 0)$$

Concavity formula

Given a parametric curve $\big(x(t),\, y(t)\big)$, its concavity satisfies the formula:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'(t)}{x'(t)}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx} = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{1}{x'(t)}$$

$t = t(x)$

$\frac{dw}{dx} = \frac{dw}{\cancel{dt}}\cdot\frac{\cancel{dt}}{dx}$

$x = f(t)$
$t = f^{-1}(t)$

$\frac{dt}{dx} = \frac{1}{\frac{d}{dt}f^{-1}(t)}$

$\frac{dt}{dx} = \frac{1}{dx/dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)\cdot\frac{1}{x'(t)} \qquad \text{(where } x'(t) \neq 0\text{)}$$

Extra - Derivation of slope and concavity formulas

For both derivations, it is necessary to view t as a function of x through the inverse parameter function. For example if $x = f(t)$ is the parametrization, then $t = f^{-1}(x)$ is the inverse parameter function.

We will need the derivative $\frac{dt}{dx}$ in terms of t. For this we use the formula for derivative of inverse functions:

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Given all this, both formulas are simple applications of the chain rule.

For the slope:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \qquad \ggg \qquad y'(t)\cdot\frac{1}{dx/dt}$$

$$\ggg \qquad \frac{y'(t)}{x'(t)}$$

For the concavity:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad \ggg \qquad \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$\ggg \qquad \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)\cdot\frac{1}{x'(t)}$$

(In the second step we inserted the formula for $\frac{dy}{dx}$ from the slope.)

Pure vertical, Pure horizontal movement

In view of the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, we see:

- Pure vertical: when $x'(t) = 0$ and yet $y'(t) \neq 0$
- Pure horizontal: when $y'(t) = 0$ and yet $x'(t) \neq 0$

When $x'(t_0) = y'(t_0) = 0$ for the same $t = t_0$, we have a **stationary point**, which might subsequently progress into pure vertical, pure horizontal, or neither.

04 Illustration

Example - Tangent to a cycloid

Find the tangent line (described parametrically) to the cycloid $(4t - 4\sin t,\, 4 - 4\cos t)$ when $t = \pi/4$.

Solution

(1) Compute x' and y'.

Find $x'(t)$:

$$x(t) = 4t - 4\sin t \quad \gg\gg \quad x'(t) = 4 - 4\cos t$$

Find $y'(t)$:

$$y(t) = 4 - 4\cos t \quad \gg\gg \quad y'(t) = 4\sin t$$

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(2) Plug in $t = \pi/4$:

$$x'(\pi/4) \quad \gg\gg \quad 4 - 4\cos(\pi/4) \quad \gg\gg \quad 4 - 2\sqrt{2}$$

Plug in $t = \pi/4$:

$$y'(\pi/4) \quad \gg\gg \quad 4\sin(\pi/4) \quad \gg\gg \quad 2\sqrt{2}$$

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(3) Apply formula: $\dfrac{dy}{dx} = \dfrac{y'}{x'}$:

Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \qquad \gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

Simplify:

$$\gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}}$$

$$\gg\gg \qquad \frac{8\sqrt{2} + 8}{16 - 8} \quad \gg\gg \quad \sqrt{2} + 1$$

So:

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2} + 1$$

This is the slope m for our line.

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(4) Need the point P for our line. Find (x, y) at $t = \pi/4$.

Plug $t = \pi/4$ into parametric formulas:

Cycloid: $(x,y) = (4t - 4\sin t,\, 4 - 4\cos t)$

Find tangent line when $t = \pi/4$.

Method: point-slope line

Find $m = \dfrac{dy}{dx}$, find (x_0, y_0) at $t = \pi/4$.

① $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{+4\sin t}{4 - 4\cos t}$, so $\left.\dfrac{dy}{dx}\right|_{t=\pi/4} = \dfrac{2\sqrt{2}}{4 - 2\sqrt{2}}$

$\dfrac{2\sqrt{2}}{4-2\sqrt{2}} \cdot \dfrac{4+2\sqrt{2}}{4+2\sqrt{2}} = \dfrac{8\sqrt{2}+8}{16-8} = \sqrt{2}+1 = m$

② $(x_0, y_0) = (\pi - 2\sqrt{2},\, 4 - 2\sqrt{2})$

pt-slope: $y = m(x - x_0) + y_0$

$\rightsquigarrow$ $y = (\sqrt{2}+1)(x - \pi + 2\sqrt{2}) + 4 - 2\sqrt{2}$

Parametric Line version:

$x = a + rt, \quad y = b + st$

$a = x(\pi/4) = \pi - 2\sqrt{2}$ | $r = x'(\pi/4) = 4 - 2\sqrt{2}$

$b = y(\pi/4) = 4 - 2\sqrt{2}$ | $s = y'(\pi/4) = 2\sqrt{2}$

$x(t) = \pi - 2\sqrt{2} + (4 - 2\sqrt{2})t$

$y(t) = 4 - 2\sqrt{2} + (2\sqrt{2})t$

$$\big(x(t),\, y(t)\big)\Big|_{t=\pi/4} \quad \gg\!\gg \quad \Big(4\frac{\pi}{4} - 4\sin(\pi/4),\ 4 - 4\cos(\pi/4)\Big)$$

$$\gg\!\gg \quad \Big(\pi - 2\sqrt{2}, 4 - 2\sqrt{2}\Big)$$

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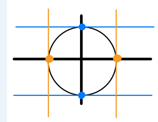
(5) Point-slope formulation of tangent line:

$$x = a + t, \quad y = b + mt$$ $\longrightarrow$ add speed $= \sqrt{1+m^2}$

Inserting our data:

$$x \ = \ (\pi - 2\sqrt{2}) + t, \qquad y \ = \ (4 - 2\sqrt{2}) + (\sqrt{2}+1)t$$

⁝☰ Example - Vertical and horizontal tangents of the circle



$(x,y) = (\cos t, \sin t)$

$x' = -\sin t$
$y' = +\cos t$

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

verticals $x' = 0, \quad y' \neq 0$

$x' = -\sin t = 0$ @ $t = 0, \pi$

Check: $y'(0) = 1, \quad y'(\pi) = -1$
$\neq 0$ ✓

$(x,y) = (1,0)$ @ $t = 0$
$= (-1,0)$ @ $t = \pi$

horizontals: $y' = 0, \quad x' \neq 0$

$y' = \cos t = 0$ @ $t = \frac{\pi}{2}, \frac{3\pi}{2}$

Check: $x'(\pi/2) = -1, \quad x'(\frac{3\pi}{2}) = +1$
$\neq 0$ ✓

$(x,y) = (0,1)$ @ $t = \pi/2$
$(0,-1)$ @ $t = \frac{3\pi}{2}$

Solution

(1) For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$x'(t) = -\sin t,$$

$$x'(t) = 0 \qquad \gg\!\gg \qquad -\sin t = 0$$

$$\gg\!\gg \qquad t = 0,\ \pi$$

The moving point is at $(1,0)$ when $t = 0$, and at $(-1,0)$ when $t = \pi$.

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(2) For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$y'(t) = \cos t,$$

$$y'(t) = 0 \qquad \gg\!\gg \qquad \cos t = 0$$

$$\gg\!\gg \qquad t = \frac{\pi}{2},\ \frac{3\pi}{2}$$

The moving point is at $(0,1)$ when $t = \pi/2$, and at $(0,-1)$ when $t = 3\pi/2$.

⁝☰ Example - Finding the point with specified slope

Consider the parametric curve given by $(x, y) = (t^2, t^3)$. Find the point where the slope of the tangent line to this curve equals 5.

Solution

(1) Compute the derivatives:

$$x'(t) \;=\; 2t, \qquad y'(t) \;=\; 3t^2$$

Therefore the slope of the tangent line, in terms of t:

$$m = \frac{dy}{dx} \;=\; \frac{y'(t)}{x'(t)}$$

$$\gg\gg \quad \frac{3t^2}{2t} \quad \gg\gg \quad \frac{3}{2}t$$

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(2) Set up equation:

$$m \;=\; 5$$

$$\frac{3}{2}t \;=\; 5$$

Solve. Obtain $t = \frac{10}{3}$.

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(3) Find the point:

$$(x, y)\Big|_{t=10/3} \quad \gg\gg \quad \left(\frac{100}{9}, \frac{1000}{27}\right)$$

05 Theory - Arclength

$ds = \sqrt{x'^2 + y'^2}\, dt$

Arclength formula

The **arclength** of a parametric curve with coordinate functions $x(t)$ and $y(t)$ is:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

This formula assumes the curve is traversed one time as t increases from a to b.

Counts total traversal

This formula applies when the curve image is traversed *one time* by the moving point.

Sometimes a parametric curve traverses its image with repetitions. The arclength formula would add length from each repetition!

Extra - Derivation of arclength formula

The arclength of a parametric curve is calculated by integrating the infinitesimal arc element:

$$ds = \sqrt{dx^2 + dy^2}$$

$$L = \int_a^b ds$$

In order to integrate ds in the t variable, as we must for parametric curves, we convert ds to a function of t:

$$ds = \sqrt{dx^2 + dy^2} \qquad \ggg \qquad \sqrt{\frac{1}{dt^2} \cdot (dx^2 + dy^2) \cdot dt^2}$$

$$\ggg \qquad \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} \cdot \sqrt{dt^2} \qquad \ggg \qquad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\ggg \qquad ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

So we obtain $ds = \sqrt{(x')^2 + (y')^2}\, dt$ and the arclength formula follows from this:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

06 Illustration

Example - Perimeter of a circles

(1) The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 = (-R\sin t)^2 + (R\cos t)^2$$

$$\ggg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \ggg \qquad R^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = R\, dt$$

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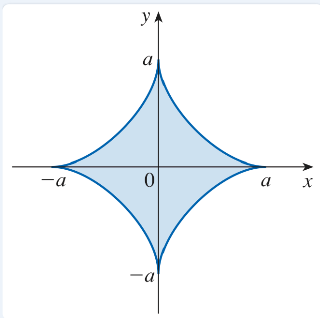
(2) Integrate around the circle:

$$\text{Perimeter} \quad = \quad \int_0^{2\pi} ds \qquad \gg\!\gg \qquad \int_0^{2\pi} R\,dt$$

$$\gg\!\gg \qquad Rt\Big|_0^{2\pi} = 2\pi R$$

Example - Perimeter of an asteroid

Find the perimeter length of the 'asteroid' given parametrically by $(x, y) = \big(a\cos^3\theta,\, a\sin^3\theta\big)$ for $a = 2$.



$(a\cos^3\theta, \quad a\sin^3\theta)$

$x' = -3a\cos^2\theta\sin\theta$

$y' = 3a\sin^2\theta\cos\theta$

$x'^2 + y'^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$

$= 9a^2\cos^2\theta\sin^2\theta\,(\cos^2 + \sin^2)$

$= 9a^2\cos^2\theta\sin^2\theta$

$\sqrt{x'^2 + y'^2} = 3a\,|\cos\theta\sin\theta|$

$L = \int_0^{2\pi} 3a\,\underbrace{|\cos\theta\sin\theta|}\,d\theta \quad = \int_0^{2\pi} ds$

$= \begin{cases} \cos\theta\sin\theta & 0 \le \theta \le \pi/2,\ \pi \le \theta \le 3\pi/2 \\ -\cos\theta\sin\theta & \pi/2 \le \theta \le \pi,\ 3\pi/2 \le \theta \le 2\pi \end{cases}$

$L = \int_0^{\pi/2} 3a\cos\theta\sin\theta\,d\theta + \int_{\pi/2}^{\pi} -3a\cos\theta\sin\theta\,d\theta + \int_{\pi}^{3\pi/2} 3a\cos\theta\sin\theta\,d\theta + \int_{3\pi/2}^{2\pi} -3a\cos\theta\sin\theta\,d\theta$

Just one: $\int_0^{\pi/2} 3a\cos\theta\sin\theta\,d\theta = \int_0^1 3a\,u\,du = 3a\,\frac{u^2}{2}\Big|_0^1 = \frac{3a}{2}$

$L = 4 \times \frac{3a}{2} = \boxed{6a}$

Solution

(1) Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$(x', y') = \big(3a\cos^2\theta\sin\theta,\, 3a\sin^2\theta\cos\theta\big)$$

---

(2) Compute the infinitesimal arc element.

$$(x')^2 + (y')^2 \quad \gg\!\gg \quad 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$\gg\!\gg \qquad 9a^2\sin^2\theta\cos^2\theta\,\big(\cos^2\theta + \sin^2\theta\big)$$

$$\gg\!\gg \qquad 9a^2\sin^2\theta\cos^2\theta$$

Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + y')^2}\,d\theta$$

$$\gg\!\gg \qquad \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta$$

$$\gg\!\gg \qquad ds = 3a\big|\sin\theta\cos\theta\big|\,d\theta$$

---

(3) Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a \sin\theta \cos\theta$ is *positive*. So we can drop the absolute value and integrate directly.

⚠ Absolute values matter!

If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a \sin\theta \cos\theta$ really does change sign.

To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

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(4) Integrate the arc element:

$$\int_0^{\pi/2} ds \quad \gg\gg \quad \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$\gg\gg \quad 3a \int_{u=0}^{1} u \, du \qquad (u = \sin\theta)$$

$$\gg\gg \quad 3a \frac{u^2}{2}\bigg|_0^1 \quad \gg\gg \quad \frac{3a}{2}$$

Finally, multiply by 4 to get the total perimeter: $L = 6a$