W14 - Notes

More calculus with parametric curves

07 Theory - Distance, speed

Distance function

The **distance function** $s(t)$ returns the total distance traveled by the particle from a chosen starting time t_0 up to the (input) time t:

$$s(t) \;=\; \int_{t_0}^{t} ds \quad=\quad \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

We need the dummy variable u so that the integration process does not conflict with t in the upper bound.

Speed function

The **speed** of a moving particle is the *rate of change of distance*:

$$v(t) \;=\; s'(t) \quad=\quad \sqrt{x'(t)^2 + y'(t)^2}$$

This formula can be explained in either of two ways:

1. Apply the Fundamental Theorem of Calculus to the integral formula for $s(t)$.
2. Consider $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ for a small change dt: so the *rate of change* of arclength is $\frac{ds}{dt}$, in other words $s'(t)$.

08 Illustration

Example - Speed, distance, displacement

The parametric curve $\left(t, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)

(1) Compute *derivatives*:

$$(x',\, y') = \left(1,\, t^{1/2}\right)$$

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(2) Now compute the *speed*.

Find sum of squares:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t$$

Get the speed function:

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{1+t}$$

(b) *Distance traveled* by using *speed*.

(1) Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

$$\int_0^t \sqrt{1+u}\, du \qquad w = 1+u \quad dw = du$$

$$\int_1^{1+t} \sqrt{w}\, dw$$

$$= \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}1^{3/2}$$

plug $t=8$:

$$\frac{2}{3}(27) - \frac{2}{3}(1)$$

$$= \boxed{\frac{52}{3}}$$

---

(2) Integrate.

Substitute $w = 1+u$ and $dw = du$.

New bounds are 1 and $1+t$.

Calculate:

$$\gg\!\gg \quad \int_1^{1+t} \sqrt{w}\, dw$$

$$\gg\!\gg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \gg\!\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

---

(3) Insert $t = 8$ for the answer.

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\!\gg \quad \frac{2}{3}(27 - 1) \quad \gg\!\gg \quad \frac{52}{3}$$

This is our final answer.

$(x_0, y_0) = pt_0$ $(x_1, y_1) = pt_1$

(c)

(1) Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Pythagorean formula for distance between given points.

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(2) Compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \qquad \gg\gg \qquad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=0} \qquad \gg\gg \qquad (0,0)$$

For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \gg\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=8}$$

$$\gg\gg \quad \left(8, \frac{2}{3}8^{3/2}\right) \quad \gg\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

---

(3) Plug points into distance formula.

Insert $(0,0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \qquad \gg\gg \qquad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\gg \qquad \frac{\sqrt{2624}}{3}$$

This is our final answer.

09 Theory - Surface area of revolutions

Recall $A = \int_a^b 2\pi R \sqrt{1+f'^2}\,dx = \int 2\pi R\,ds$

$= \sqrt{x'^2 + y'^2}\,dt$ $\shortparallel$ ds

Surface area of a surface of revolution: thin bands

Suppose a parametric curve $\big(x(t),\, y(t)\big)$ is revolved around the x-axis or the y-axis.

The surface area is:

$$A \quad = \quad \int_a^b 2\pi R(t)\,\sqrt{(x')^2 + (y')^2}\,dt$$

The radius $R(t)$ should be the distance to the axis:

$$\begin{aligned} R(t) &= y(t) \qquad \text{revolution about } x\text{-axis} \\ R(t) &= x(t) \qquad \text{revolution about } y\text{-axis} \end{aligned}$$

This formulas adds the areas of thin bands, but the bands are demarcated using parametric functions instead of input values of a graphed function.

The formula assumes that the curve is traversed one time as t increases from a to b.

10 Illustration

Example - Surface of revolution - parametric circle

(1) By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

The parametrization of the unit upper semicircle is: $(x, y) = (\cos t, \sin t)$.

The derivative is: $(x', y') = (-\sin t, \cos t)$.

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(2) Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$$\gg\!\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \quad \gg\!\gg \quad dt$$

---

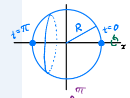
(3) Now for R we choose $R = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^{\pi} 2\pi \sin t\, dt \quad \gg\!\gg \quad -2\pi \cos t\Big|_0^{\pi} \quad \gg\!\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

Surface area of a sphere:



$(x,y) = (R\cos t, R\sin t)$

$x' = -R\sin t$
$y' = R\cos t$

$A = \int_{t=0}^{\pi} 2\pi R \sqrt{x'^2 + y'^2}\, dt$

$x'^2 + y'^2 = R^2$
$\sqrt{x'^2 + y'^2} = R$

$R(t) = y(t) = R\sin t$

$\rightsquigarrow A = \int_0^{\pi} 2\pi R \sin t \, R\, dt = 2\pi R^2 \int_0^{\pi} \sin t\, dt$
$= \boxed{4\pi R^2}$

Example - Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3, t^2 - 1)$ for $0 \le t \le 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$(x,y) = (t^3, t^2-1)$ around x-axis
$t \in [0,1]$

$A = \int_0^1 2\pi (t^2-1) \sqrt{9t^4 + 4t^2}\, dt$
$= \int_0^1 2\pi (t^2-1) t \sqrt{9t^2+4}\, dt$

Method 1:
$\sqrt{9t^2+4} = 3\sqrt{t^2 + 4/9}$ $\quad t = \frac{2}{3}\tan\theta$
$= 3\sqrt{\frac{4}{9}\tan^2\theta + \frac{4}{9}} = 2\sqrt{\sec^2\theta}$
$= 2\sec\theta \ldots.$

Method 2: $u = 9t^2 + 4$ $\quad t^2 = \frac{u-4}{9}$
$du = 18t\, dt$ $\quad t^2 - 1 = \frac{u-13}{9}$

$\int 2\pi \left(\frac{u-13}{9}\right) \frac{1}{18} \sqrt{u}\, du$

$$ds = \sqrt{(x')^2 + (y')^2} \quad \ggg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \ggg \quad \sqrt{9t^4 + 4t^2}$$

$$\ggg \quad \sqrt{t^2(9t^2+4)} \quad \ggg \quad t\sqrt{9t^2+4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \ggg \quad \int_0^1 2\pi(t^2-1)t\sqrt{9t^2+4}\, dt$$

Polar curves

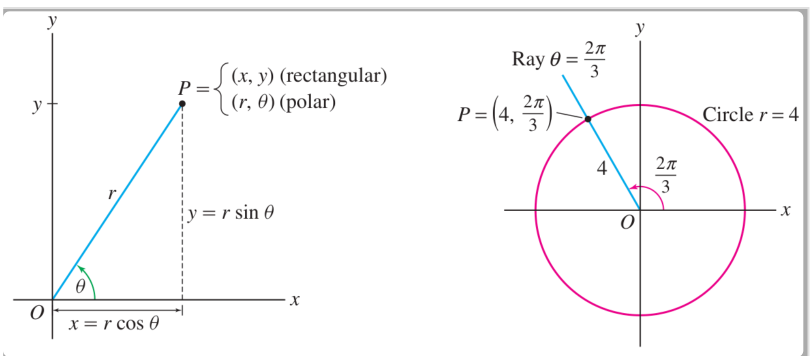
Videos

Videos, Organic Chemistry Tutor

- Polar coordinates intro
- Graphing polar curves

01 Theory - Polar points, polar curves

Polar coordinates are pairs of numbers (r, θ) which identify points in the plane in terms of *distance to origin* and *angle from $+x$-axis*:



Converting Polar $\leftrightarrow$ Cartesian

Polar $\to$ Cartesian	Cartesian $\to$ Polar
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x} \quad (x \neq 0)$

Polar coordinates have *many redundancies*: unlike Cartesian which are unique!

- For example: $(r, \theta) = (r, \theta + 2\pi)$
  - And therefore also $(r, \theta) = (r, \theta - 2\pi)$ (negative θ can happen)

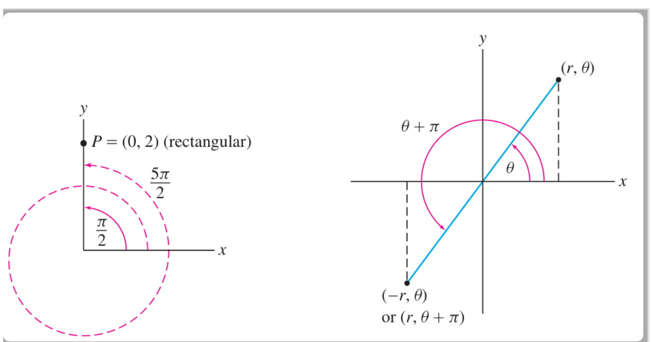
- For example: $(-r, \theta) = (r, \theta + \pi)$ for every r, θ
- For example: $(0, \theta) = (0, 0)$ for any θ

Polar coordinates *cannot be added*: they are not vector components!

- For example $(5, \pi/3) + (2, \pi/6) \neq (7, \pi/2)$
- Whereas Cartesian coordinates can be added: $(1, 4) + (2, -2) = (3, 2)$

⚠ The transition formulas Cartesian → Polar require careful choice of θ.

- The standard definition of $\tan^{-1}\left(\frac{y}{x}\right)$ sometimes gives *wrong θ*
  - This is because it uses the restricted domain $\theta \in (-\pi/2, \pi/2)$; the polar interpretation is: only points in Quadrant I and Quadrant IV (SAFE QUADRANTS)
- Therefore: *check signs* of x and y to see *which quadrant*, maybe need π-correction!
  - Quadrant I or IV: polar angle is $\tan^{-1}\left(\frac{y}{x}\right)$
  - Quadrant II or III: polar angle is $\tan^{-1}\left(\frac{y}{x}\right) + \pi$



Equations (as well as points) can also be converted to polar.

For Cartesian → Polar, look for cancellation from $\cos^2\theta + \sin^2\theta = 1$.

For Polar → Cartesian, try to keep θ inside of trig functions.

- For example:

Q: convert "$r = \sin^2\theta$" to Cartesian eqn.

$$r = \sin^2\theta \quad \gg\!\gg \quad \sqrt{x^2+y^2} = \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2$$

A:
use $r = \sqrt{x^2+y^2}$
(DON'T use $\theta = \tan^{-1}(y/x)$)
DO use: $y = r\sin\theta$
$\leadsto \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$
$\leadsto \sin^2\theta = \frac{y^2}{x^2+y^2}$

$\sqrt{x^2+y^2} = \frac{y^2}{x^2+y^2}$

$\boxed{y^2 = (x^2+y^2)^{3/2}}$

02 Illustration

☰ Example - Converting to polar: π-correction

Compute the polar coordinates of $\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \gg\gg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \gg\gg \quad -\pi/3$$

This angle is in Quadrant IV. We *add* π to get the polar angle in Quadrant II:

$$\theta = \pi - \pi/3 \quad \gg\gg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r, \theta) = (1, 2\pi/3)$.

For $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV. (No extra π needed.)

Next compute:

$$\tan^{-1}\left(\frac{-\sqrt{2}/2}{+\sqrt{2}/2}\right) \quad \gg\gg \quad \tan^{-1}(-1) \quad \gg\gg \quad -\pi/4$$

So the point in polar is $(1, -\pi/4)$.

$\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$

Quadrant II unsafe

So: $r = \sqrt{x^2+y^2}$

$= \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$ ✓ (unit circle)

$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$

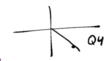
$= \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) + \pi$

$= \tan^{-1}\left(-\sqrt{3}\right) + \pi$

$= -\frac{\pi}{3} + \pi = 2\pi/3$

So $\boxed{(r,\theta) = (1, 2\pi/3)}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$



Q4 - safe

$r = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{2}/2}{+\sqrt{2}/2}\right) = \tan^{-1}(-1)$

$= -\pi/4$

$\boxed{(r,\theta) = (1, -\pi/4) = (1, 7\pi/4)}$

☰ Example - Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y-3)^2 = 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \quad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

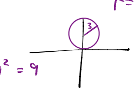
$$\gg\gg \quad r^2 - 6r\sin\theta = 0 \quad \gg\gg \quad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.

$x^2 + (y-3)^2 = 9$

circle @ $(h,k) = (0,3)$
$r = 3$

$x = r\cos\theta$
$y = r\sin\theta$



$r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$

$r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$

$\leadsto r^2 - 6r\sin\theta = 0$

$\leadsto r(r - 6\cdot\sin\theta) = 0$

$r = 0$ OR $r = 6\sin\theta$

03 Theory - Polar limaçons

To draw the polar graph of some function, it can help to first draw the Cartesian graph of the function. (In other words, set $y = r$ and $x = \theta$, and draw the usual graph.) By tracing through the points on the Cartesian graph, one can visualize the trajectory of the polar graph.

This Cartesian graph may be called a **graphing tool** for the polar graph.

OR $\sin\theta$

scale up; doesn't change shape (only size)

A limaçon is the polar graph of $r(\theta) \;=\; a + b\cos\theta$. $\rightsquigarrow a\left(1 + \frac{b}{a}\cos\theta\right)$

Shape same as $1 + \frac{b}{a}\cos\theta$

Say have $1+2\cos\theta$.

What is shape of $5(1+2\cos\theta)$

i.e. of $5+10\cos\theta$?

$r(\theta) \longrightarrow r = 5\cdot r(\theta)$

Graph of $r(\theta - \frac{\pi}{2})$ is same as graph of $r(\theta)$ except... ROTATED by $\frac{\pi}{2}$ counterclockwise (ccw)

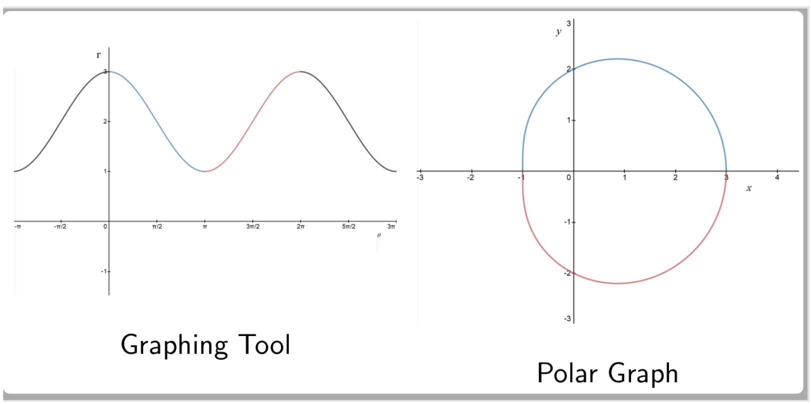
$\left(r(\theta + \frac{\pi}{3})\text{: rotate } \frac{\pi}{3} \text{ clockwise}\right)$

Any limaçon shape can be obtained by adjusting c in this function (and rescaling):

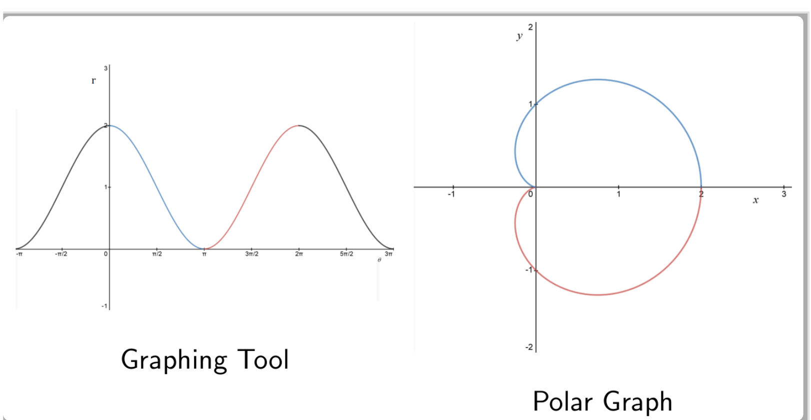
$$r = 1 + c\cos\theta$$

Limaçon satisfying $r(\theta) = 1$: unit circle.

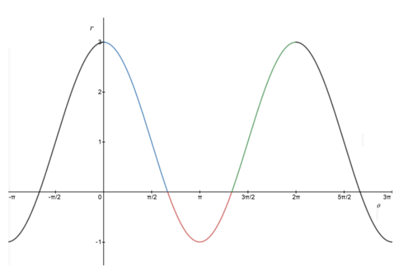
Limaçon satisfying $r(\theta) = 2 + \cos\theta$: 'outer loop' circle with 'dimple':



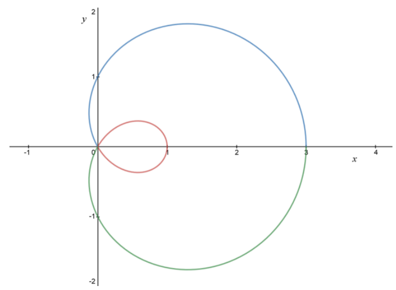
Limaçon satisfying $r(\theta) = 1 + \cos\theta$: 'cardioid' = 'outer loop' circle with 'dimple' that creates a cusp:



Limaçon satisfying $r(\theta) = 1 + 2\cos\theta$: 'dimple' pushes past cusp to create 'inner loop':

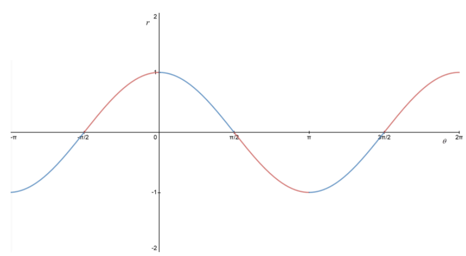


Graphing Tool

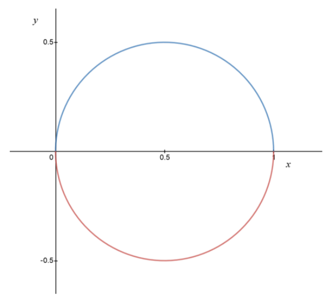


Polar Graph

Limaçon satisfying $r(\theta) = \cos\theta$: ‘inner loop’ only, no outer loop exists:



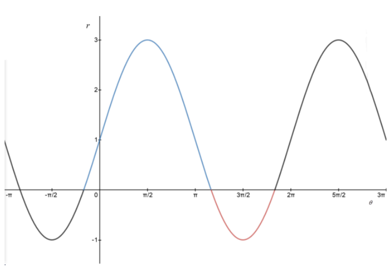
Graphing Tool



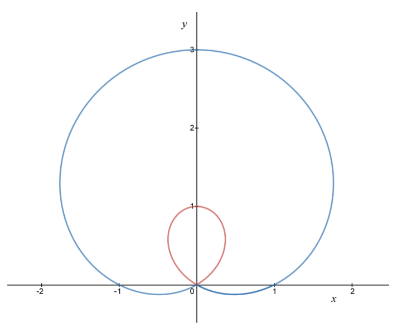
Polar Graph

$\cos(\theta - \frac{\pi}{2})$
$\parallel$

Limaçon satisfying $r(\theta) = 1 + 2\sin\theta$: ‘inner loop’ and ‘outer loop’ and rotated ↺ $90°$:

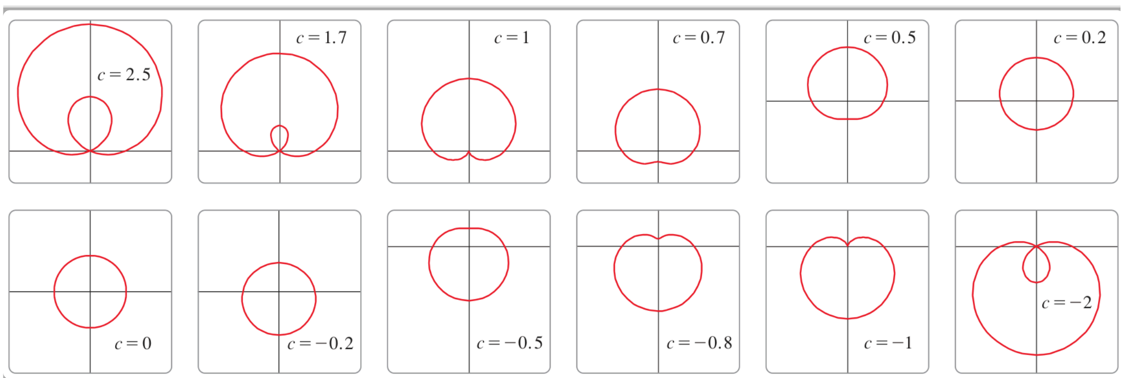


Graphing Tool



Polar Graph

Transitions between limaçon types, $r(\theta) \;=\; 1 + c\sin\theta$:



Notice the transition points at $|c| = 0.5$ and $|c| = 1$:

The *flat spot* occurs when $c = \pm 0.5$

- Smaller c gives *convex shape*

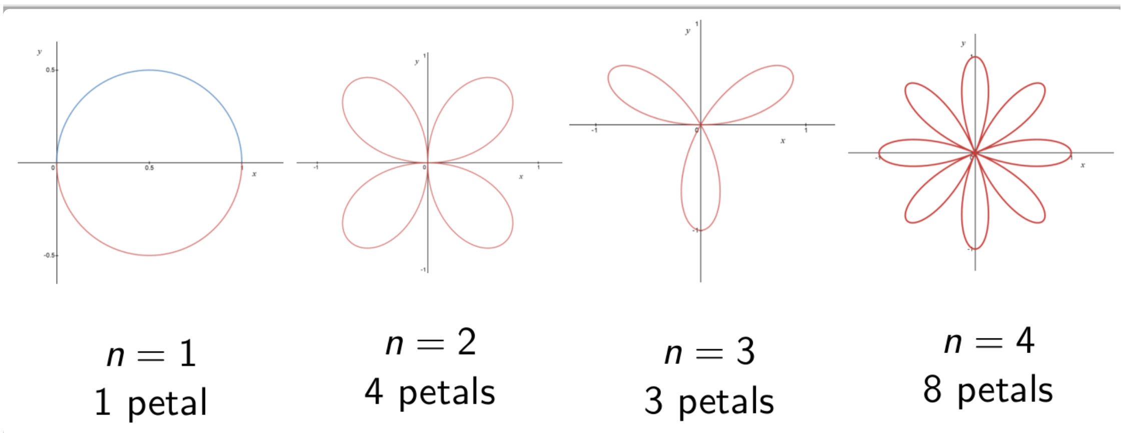
The *cusp* occurs when $c = \pm 1$

- Smaller c gives *dimple* (assuming $|c| > 0.5$)
- Larger c gives *inner loop*

04 Theory - Polar roses

Roses are polar graphs of this form:

$$r = \cos(\theta), \qquad r = \sin(2\theta), \qquad r = \sin(3\theta), \qquad r = \overset{\cos(4\theta)}{\cancel{\sin(4\theta)}}$$



The pattern of petals:

- $n = 2k$ (even): obtain $2n$ petals
  - These petals traversed *once*
- $n = 2k + 1$ (odd): obtain n petals
  - These petals traversed *twice*
- Either way: total-petal-traversals: always $2n$