W15 - Notes

Calculus with polar curves

05 Theory - Polar tangent lines, arclength

Polar arclength formula

The arclength of the polar graph of $r(\theta)$, for $\theta \in [\alpha, \beta]$:

$$L = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\, d\theta$$

To derive this formula, *convert to Cartesian* with parameter θ:

$$r = r(\theta) \quad \gg\gg \quad (x, y) = (r\cos\theta,\, r\sin\theta)$$

From here you can apply the familiar arclength formula with θ in the place of t.

Extra - Derivation of polar arclength formula

Let $r = r(\theta)$ and convert to parametric Cartesian, so:

$$\begin{aligned} x(\theta) &= r(\theta)\cos\theta \\ y(\theta) &= r(\theta)\sin\theta \end{aligned}$$

Then:

$$ds = \sqrt{(x')^2 + (y')^2}\, d\theta$$

$$\begin{aligned} x' = (r\cos\theta)' &\quad \gg\gg \quad r'\cos\theta - r\sin\theta \\ y' = (r\sin\theta)' &\quad \gg\gg \quad r'\sin\theta + r\cos\theta \end{aligned}$$

Therefore:

$$\begin{aligned} (x')^2 + (y')^2 \quad \gg\gg \quad & r'^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta \\ & + r'^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta \\ & = r'^2 + r^2 \end{aligned}$$

Therefore:

$$ds = \sqrt{(x')^2 + (y')^2}\, d\theta \quad \gg\gg \quad \sqrt{r'^2 + r^2}\, d\theta$$

06 Illustration

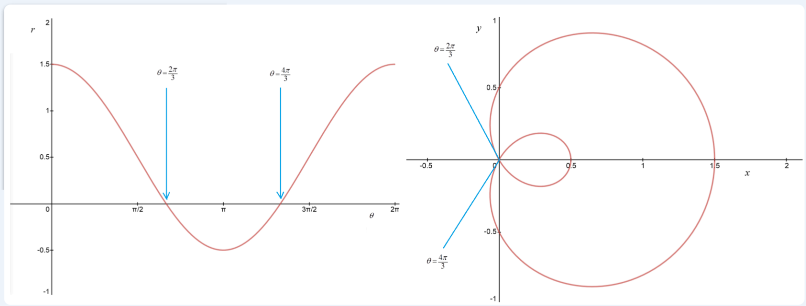
Example - Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos\theta$.

How long is the inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

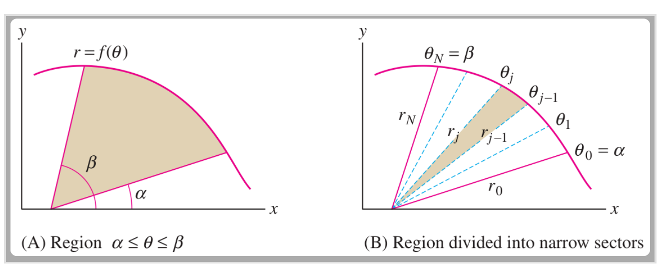
$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(-\sin\theta)^2 + \left(\frac{1}{2} + \cos\theta\right)^2}\, d\theta \quad \gg\gg \quad \int_{2\pi/3}^{4\pi/3} \sqrt{5/4 + \cos\theta}\, d\theta$$

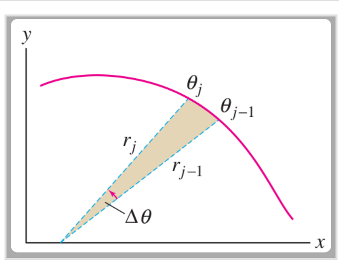
07 Theory - Polar area

Sectorial area from polar curve

$$A = \int_\alpha^\beta \frac{1}{2} r(\theta)^2\, d\theta$$

The "area under the curve" concept for graphs of functions in Cartesian coordinates translates to a "sectorial area" concept for polar graphs. To compute this area using an integral, we divide the region into Riemann sums of small sector slices.

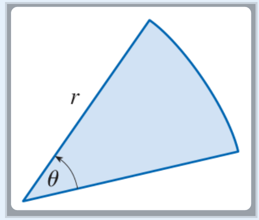




To obtain a formula for the whole area, we need a formula for the area of each sector slice.

Area of sector slice

Let us verify that the area of a sector slice is $\frac{1}{2}r^2\theta$.



Take the angle θ in radians and divide by 2π to get the *fraction of the whole disk*.

Then multiply this fraction by πr^2 (whole disk area) to get the *area of the sector slice*.

$$\left(\frac{\theta}{2\pi}\right)(\pi r^2) \quad \ggg \quad \frac{1}{2}r^2\theta$$

Now use $d\theta$ and $r(\theta)$ for an infinitesimal sector slice, and integrate these to get the total area formula:

$$A = \int_\alpha^\beta \frac{1}{2}r(\theta)^2\, d\theta$$

One easily verifies this formula for a circle.

Let $r(\theta) = R$ be a constant. Then:

$$\text{Area of circle} = \int_0^{2\pi} \frac{1}{2}R^2\, d\theta \quad \ggg \quad \frac{1}{2}R^2\theta\Big|_0^{2\pi} \quad \ggg \quad R^2\pi$$

The sectorial area *between curves*:

Sectorial area between polar curves

$$A = \int_\alpha^\beta \frac{1}{2}\left(r_1(\theta)^2 - r_0(\theta)^2\right) d\theta$$

Subtract *after* squaring, not before!

This aspect is *not* similar to the Cartesian version: $\int f - g\, dx$

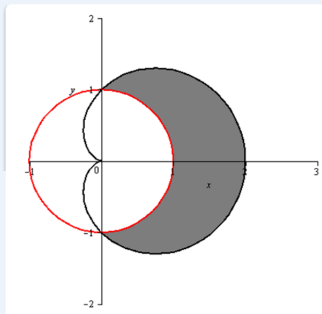
08 Illustration

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(\theta) = 1$ and the limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution

First draw the region:



The two curves intersect at $\theta = \pm\frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$:

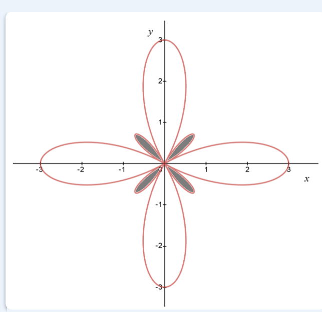
$$A = \int_a^b \frac{1}{2}(r_1^2 - r_0^2)\,d\theta \quad \ggg \quad \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - 1^2\Big)\,d\theta$$

$$\ggg \quad \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta\,d\theta \quad \ggg \quad \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{4}\big(1+\cos(2\theta)\big)\,d\theta$$

$$\ggg \quad \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \quad \ggg \quad \boxed{2 + \frac{\pi}{4}}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

Find bounds for one small loop. Lower left loop occurs first. This loop is when $1 + 2\cos(4\theta) \le 0$.

$$1 + 2\cos(4\theta) \le 0 \qquad \gg\gg \qquad \cos(4\theta) \le -\frac{1}{2}$$

$$\gg\gg \qquad \frac{2\pi}{3} \le 4\theta \le \frac{4\pi}{3} \qquad \gg\gg \qquad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

Now set up area integral:

$$A = 4\int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta \qquad \gg\gg \qquad 4\int_{\pi/6}^{\pi/3} \frac{1}{2}\big(1 + 2\cos(4\theta)\big)^2 \, d\theta$$

$$\gg\gg \qquad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4\cos^2(4\theta) \, d\theta$$

Power-to-frequency conversion: $\cos^2 A \rightsquigarrow \frac{1}{2}(1 + \cos(2A))$ with $A = 4\theta$:

$$\gg\gg \qquad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4 \cdot \frac{1}{2}\big(1 + \cos(8\theta)\big) \, d\theta$$

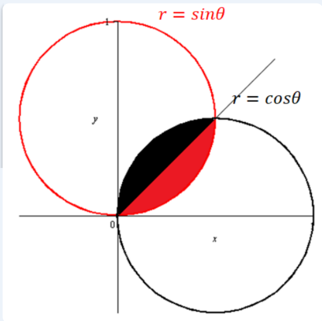
$$\gg\gg \qquad 6\theta + 2\sin(4\theta) + \frac{1}{4}\sin(8\theta)\Big|_{\pi/6}^{\pi/3} \qquad \gg\gg \qquad \pi - \frac{3\sqrt{3}}{2}$$

☰ Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta) = \sin\theta$ and the other is given by $r(\theta) = \cos\theta$.

Solution

Drawing of the overlap:



Notice: total overlap area = 2× area of red region. Bounds for red region: $0 \le \theta \le \frac{\pi}{4}$.

Area formula applied to $r(\theta) = \sin\theta$:

$$A \;=\; 2\int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta \qquad \gg\gg \qquad 2\int_{0}^{\pi/4} \frac{1}{2}\sin^2\theta \, d\theta$$

Power-to-frequency: $\sin^2\theta \rightsquigarrow \frac{1}{2}\big(1 - \cos(2\theta)\big)$:

$$\gg\gg \quad 2\int_0^{\pi/4} \frac{1}{4}\big(1-\cos(2\theta)\big)\,d\theta$$

$$\gg\gg \quad \frac{2}{4}\theta - \frac{2}{8}\sin(2\theta)\Big|_0^{\pi/4} \quad \gg\gg \quad \frac{\pi}{8} - \frac{1}{4}$$

Complex algebra

Videos

Videos, Organic Chemistry Tutor

- Complex numbers basics

01 Theory - Complex arithmetic

The complex numbers $\mathbb{C}$ are sums of real and imaginary numbers. Every complex number can be written uniquely in 'Cartesian' form:

$$z = a + bi, \qquad a, b \in \mathbb{R}$$

To add, subtract, scale, and multiply complex numbers, treat 'i' like a constant.

Simplify the result using $i^2 = -1$.

For example:

$$(1+3i)(2-2i) \quad \gg\gg \quad 2 - 2i + 6i - 6i^2$$

$$\gg\gg \quad 2 + 4i - 6(-1) \quad \gg\gg \quad 8 + 4i$$

$(1+i)^2 = 2i$ by:

$(1+i)(1+i) = 1 + i + i + i^2$ ($i^2 \to -1$)

$= 0 + 2i = 2i$

$(1+2i)(1+2i) = 1 + 2i + 2i + 4i^2$

$= 1 - 4 + 4i = -3 + 4i$

$(1+i)(1-i) = 1 + i - i - i^2$

$= 1 - -1 = +2$

$\Rightarrow \frac{2}{1+i} = 1 - i \quad \Rightarrow \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$

-OR- "$(1+i)\cdot\left(\frac{1}{2} - \frac{1}{2}i\right) = 1$"

so $\frac{1}{2} - \frac{1}{2}i$ is ~~inverse~~ reciprocal of $1+i$

$(2+3i)(2-3i) = 4 + 6i - 6i - 9i^2$

$= 13$

so $\frac{1}{2+3i} = \frac{2}{13} - \frac{3}{13}i$

Notice: cplx conjugate

1) $(a+bi)(a-bi)$ is pure real number

2) $(a+bi)(a-bi) = a^2 + b^2 > 0$ (=0 only when $a, b = 0$)

write $\overline{a+bi} = a - bi$

so $\overline{2-3i} = 2 + 3i$

Complex conjugate

Every complex number has a **complex conjugate**:

$$z = a + bi \quad \gg\gg \quad \bar{z} = a - bi$$

For example:

$$\overline{2+5i} \;=\; 2 - 5i$$

$$\overline{2-5i} \;=\; 2 + 5i$$

In general, $\bar{\bar{z}} = z$.

Conjugates are useful mainly because they eliminate imaginary parts:

$$(2+5i)(2-5i) \quad \gg\gg \quad 4 + 25 \quad \gg\gg \quad 29$$

In general:

$$(a+bi)(a-bi) \quad \gg\gg \quad a^2 - abi + bia - b^2 i^2 \quad \gg\gg \quad a^2 + b^2 \in \mathbb{R}$$

Complex division

To divide complex numbers, use the conjugate to eliminate the imaginary part in the denominator.

For example, reciprocals:

$$\frac{1}{a+bi} \quad \gg\gg \quad \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi}$$

$$\gg\gg \quad \frac{a-bi}{a^2+b^2} \quad \gg\gg \quad \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right)i$$

More general fractions:

$$\frac{a+bi}{c+di} \quad \gg\gg \quad \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$$

$$\gg\gg \quad \frac{ac+bd+(bc-ad)i}{c^2+d^2} \quad \gg\gg \quad \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Multiplication preserves conjugation

For any $z, w \in \mathbb{C}$:

$$\overline{zw} \;=\; \bar{z}\bar{w}$$

Therefore, one can take products or conjugates in either order.

02 Illustration

Example - Complex multiplication

Compute the products:

(a) $(1-i)(4-7i)$ (b) $(2+5i)(2-5i)$

Solution

(a) $(1-i)(4-7i)$

Expand:

$$(1-i)(4-7i) \quad \gg\gg \quad 4 - 7i - 4i + 7i^2$$

Simplify i^2:

$$\gg\gg \quad 4 - 7i - 4i + 7(-1) \quad \gg\gg \quad -3 - 11i$$

(b) $(2 + 5i)(2 - 5i)$

Expand:

$$(2 + 5i)(2 - 5i) \quad \gg\gg \quad 4 - 10i + 10i - 25i^2$$

Simplify i^2:

$$\gg\gg \quad 4 - 10i + 10i - 25(-1) \quad \gg\gg \quad 29$$

$$\frac{1}{1+i}\left(\frac{1-i}{1-i}\right) = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$$

$a + bi$

Example - Complex division

Compute the following divisions of complex numbers:

(a) $\dfrac{1}{-3+i}$ (b) $\dfrac{1}{i}$ (c) $\dfrac{1}{7i}$ (d) $\dfrac{2+5i}{2-5i}$

Solution

$\frac{1}{7i} = ?$

$7i = a + bi$

$a = 0$

$b = 7$

So $\overline{7i} = a - bi = 0 - 7i = -7i$

Then: $\frac{1}{7i}\left(\frac{-7i}{-7i}\right) = \frac{-7i}{49} = -\frac{1}{7}i$

Can just use $\frac{1}{i} = -i$

$\left(\frac{1}{i}\left(\frac{-i}{-i}\right) = \frac{-i}{1}\right)$

(or: $i(-i) = 1$ so $-i$ is reciprocal)

$\frac{2+5i}{2-5i} = ?$

$\frac{2+5i}{2-5i}\left(\frac{2+5i}{2+5i}\right) = \frac{4 + 10i + 10i - 25i^2}{29}$

$= \frac{-21}{29} + \frac{20}{29}i$

(a) $\dfrac{1}{-3+i}$

Conjugate is $-3 - i$:

$$\frac{1}{-3+i} \quad \gg\gg \quad \frac{1}{-3+i} \cdot \frac{-3-i}{-3-i}$$

Simplify:

$$\gg\gg \quad \frac{-3-i}{9+1} \quad \gg\gg \quad \frac{-3}{10} + \frac{-1}{10}i$$

(b) $\dfrac{1}{i}$

Conjugate is $-i$:

$$\frac{1}{i} \quad \gg\gg \quad \frac{1}{i} \cdot \frac{-i}{-i} \quad \gg\gg \quad -i$$

(c) $\dfrac{1}{7i}$

Factor out the $1/7$:

$$\frac{1}{7i} \quad \gg\gg \quad \frac{1}{7} \cdot \frac{1}{i}$$

Use $\frac{1}{i} = -i$:

$$\gg\gg \quad \frac{1}{7} \cdot (-i) \quad \gg\gg \quad \frac{-1}{7}i$$

(d) $\dfrac{2+5i}{2-5i}$

Denominator conjugate is $2+5i$:

$$\frac{2+5i}{2-5i} \quad \gg\gg \quad \frac{2+5i}{2-5i} \cdot \frac{2+5i}{2+5i}$$

Simplify:

$$\gg\gg \quad \frac{4+20i+25i^2}{4+25} \quad \gg\gg \quad \frac{-21}{29} + \frac{20}{29}i$$

§ Polar Arclength

$$L = \int_{\alpha}^{\beta} \sqrt{r'^2 + r^2}\, d\theta \qquad \text{for } r(\theta),\ \theta \in [\alpha, \beta]$$

Pf: Convert to parametric:

$x = r\cos\theta$, $y = r\sin\theta$, $r = r(\theta)$

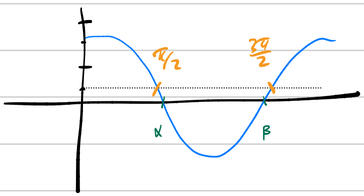
and $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$

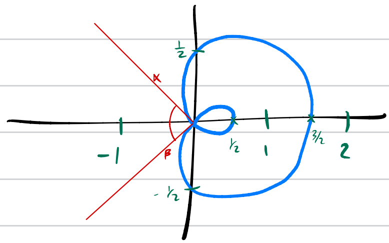
$$ds = \sqrt{x'^2 + y'^2}\, d\theta \qquad \begin{aligned} x' &= r'\cos\theta - r\sin\theta \\ y' &= r'\sin\theta + r\cos\theta \end{aligned}$$

$$\begin{aligned} x'^2 + y'^2 &= r'^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + r'^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta \\ &= r'^2 + r^2 \end{aligned}$$

Thus $ds = \sqrt{r'^2 + r^2}\, d\theta$ $\square$

Example: set up integral for length of inner loop of Limaçon: $r = \frac{1}{2} + \cos\theta$.





$\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$

$\pi/6 \quad \pi/4 \quad \pi/3$

To find α, β, solve $\frac{1}{2} + \cos\theta = 0$

$\rightsquigarrow \cos\theta = -\frac{1}{2} \rightsquigarrow \theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3} = \alpha$

$\theta = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3} = \beta$

$\theta \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

$r(\theta)^2 = \left(\frac{1}{2} + \cos\theta\right)^2 = \frac{1}{4} + \cos\theta + \cos^2\theta$

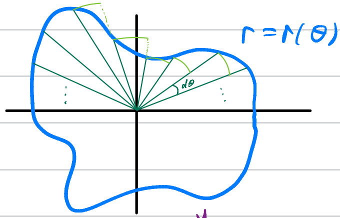
$r'(\theta)^2 = (-\sin\theta)^2 = \sin^2\theta$

$+$

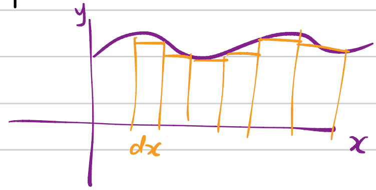
$r^2 + r'^2 = \frac{5}{4} + \cos\theta$

$$\text{ANS} = \boxed{\int_{2\pi/3}^{4\pi/3} \sqrt{\frac{5}{4} + \cos\theta}\, d\theta}$$

§ Polar Area

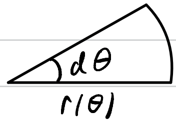


$$A = \int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta$$



Pf:

Hence need: area of:



$$A = (\% \text{ of circle})(\text{area of disk})$$

$$= \left(\frac{d\theta}{2\pi}\right)\left(\pi r(\theta)^2\right) = \frac{1}{2} r(\theta)^2 d\theta$$ ⟵ sum up these □

Also: area between polar curves:

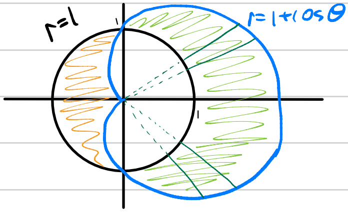
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_1(\theta)^2 - r_0(\theta)^2\right) d\theta$$

Like:

$$\int f - g \, dx$$

Example: Find area between circle $r_0(\theta) = 1$ and Limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution:



$\alpha = -\frac{\pi}{2}, \quad \beta = +\frac{\pi}{2}$

$$A = \int_{-\pi/2}^{+\pi/2} \frac{1}{2}\left((1+\cos\theta)^2 - 1^2\right) d\theta$$

"pwr. to freq. conv."

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{2}\cos^2\theta \, d\theta = \sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2}$$

$$= 2 + \frac{\pi}{4}$$

$$\int_{\pi/2}^{\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta = \int_{\pi/2}^{\pi} \frac{1}{2} + \cos\theta + \frac{1}{2}\cos^2\theta \, d\theta$$

$$= \frac{\theta}{2} + \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} - \left(\frac{\pi}{4} + 1 + \frac{\pi}{8} + 0\right) = \frac{\pi}{2} - 1 - \frac{\pi}{8} = \frac{3}{8}\pi - 1$$

quarter disk $= \frac{\pi}{4}$ $\leadsto$ $\frac{\pi}{4} - \left(\frac{3}{8}\pi - 1\right) = 1 - \frac{\pi}{8}$

double $\leadsto$ $2 - \frac{\pi}{4}$

add to prev. result: $2 - \frac{\pi}{4} + 2 + \frac{\pi}{4} = \boxed{4}$