W01 Notes

Volume using cylindrical shells

Videos

Review

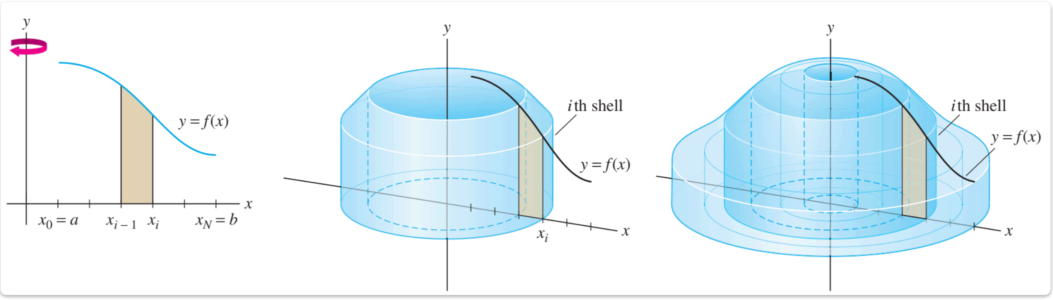
- Volume using cross-sectional area
- Disk/washer method - 01
- Disk/washer method - 02
- Disk/washer method - 03

Shells

- Shell method - 01
- Shell method - 02
- Shell method - 03

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy-plane. Rotate this about the y-axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi\, R\, h\, \Delta r$:

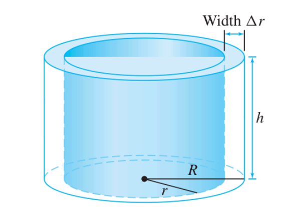


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi R h \Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \to dr$ and the number of shells becomes infinite, their total volume is given by an integral.

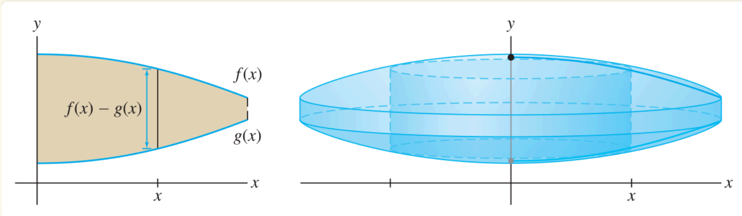
Volume by shells - general formula

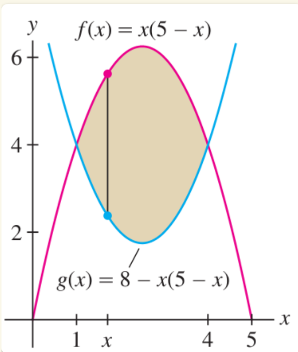
$$V=\int_a^b 2\pi Rh\,dr$$

In any concrete volume calculation, we simply interpret each factor, 'R' and 'h' and 'dr', and determine a and b in terms of the variable of integration that is set for r.

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?





02 Illustration

Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y=3x+2$, $y=6-x$, $x=0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution

(1) Cross-section region:

Bounded above-right by $y=6-x$. Bounded below-right by $y=3x+2$. These intersect at $x=1$.

Bounded left by $x=0$.

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(2) Set up integral:

Rotated around y-axis, therefore use x for integration variable (shells!). Formula:

$$V=\int_a^b 2\pi Rh\,dr$$

Domain is $[a,b]=[0,1]$.

$R(x) = x$ because shell radius is the x-distance from $x = 0$ to the shell position.

Height:

$$h(x) \quad \gg\gg \quad (6 - x) - (3x + 2)$$

$$\gg\gg \quad 4 - 4x$$

dr is limit of Δr which equals Δx here, so $dr = dx$.

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(3) Evaluate integral:

$$V = \int_a^b 2\pi Rh\, dr \quad \gg\gg \quad \int_0^1 2\pi \cdot x(4 - 4x)\, dx$$

$$\gg\gg \quad 2\pi \left(2x^2 - \frac{4x^3}{3}\right)\Big|_0^1 \quad \gg\gg \quad \boxed{\frac{4\pi}{3}}$$

Practice exercise

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y-axis. Set up an integral that gives the volume of this region using the method of shells.

Solution >

(1) Set up the integral for shells:

Integration variable: $r = x$, the distance of a shell to the y-axis.

Then $dr = dx$ and $h = \sin x$, the height of a shell.

Bounds: one hump is given by $x \in [0, \pi]$. Thus:

$$V = \int_0^\pi 2\pi x \sin x\, dx$$

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(2) Perform the integral using IBP:

Choose $u = 2\pi x$ and $v' = \sin x$ since x is A and $\sin x$ is T.

Then $u' = 2\pi$ and $v = -\cos x$.

Use IBP formula:

$$\int uv'\, dx \;=\; uv - \int u'v\, dx$$

$$\gg\gg \quad \int_0^\pi 2\pi x \sin x\, dx \quad \gg\gg \quad (2\pi x)(-\cos x)\Big|_0^\pi - \int_0^\pi 2\pi(-\cos x)\, dx$$

Compute first term:

$$\gg\gg \quad -2\pi x\cos x\Big|_0^\pi$$

$$\gg\gg \quad -2\pi(\pi)(-1) - -2\pi(0)(+1) \quad \gg\gg \quad 2\pi^2$$

Compute integral term:

$$-\int_0^\pi 2\pi(-\cos x)\,dx \quad \gg\gg \quad 2\pi\sin\Big|_0^\pi \quad \gg\gg \quad 0$$

So the answer is $2\pi^2$.

Integration by substitution (review only)

Videos

[Note: this section is non-examinable. It is included for comparison to IBP.]

- Integration by Substitution 1: $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$
- Integration by Substitution 2: $\int \frac{x^5}{(1-x^3)^3}\,dx$
- Integration by Substitution 3: $\int_0^1 x^2(1+x)^4\,dx$
- Integration by Substitution 4: $\int \frac{2x+3}{\sqrt{2x+1}}\,dx$
- Integration by Substitution 5: $\int \frac{\sin x}{\cos^3 x}\,dx$
- Integration by Substitution: Definite integrals, various examples

03 Theory

The method of **u-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

Substitution

Suppose the integral has this format, for some functions f and u:

$$\int f(u(x))\cdot u'(x)\,dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x))\cdot u'(x)\,dx \quad \gg\gg \quad \int f(u)\,du$$

The technique of u-substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx}F\big(u(x)\big) = f(u(x))\cdot u'(x)$$

Here we let $F' = f$. Thus $\int f(x)\,dx = F(x) + C$ for some C.

Now, if we *integrate both sides* of this equation, we find:

$$F\big(u(x)\big) = \int f(u(x))\cdot u'(x)\,dx$$

And of course $F(u) = \int f(u)\,du - C$.

Extra - Full explanation of u-substitution

(1) Chain rule for derivatives:

Let $F(x)$ be a function and $F' = f$ its derivative. Let $u(x)$ be another function.

Using primes:

$$\big(F \circ u\big)' = (F' \circ u) \cdot u'$$

Using differentials:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

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(2) Integrate both sides:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x) \quad \overset{\int}{\ggg} \quad \int \frac{d}{dx} F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)$$

$$\overset{\text{FTC}}{\ggg} \quad F(u(x)) = \int f(u(x)) \cdot u'(x)$$

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(3) Introduce 'variable' u from the u-format of the integral:

Treating u as a variable, the definition of F gives:

$$F(u) \;=\; \int f(u)\,du + C$$

Set the 'variable' u to the 'function' u output:

$$F(u)\,\Big|_{u=u(x)} \;=\; F(u(x))$$

Combine these:

$$F(u(x)) = F(u)\,\Big|_{u=u(x)}$$

$$= \int f(u)\,du\,\Big|_{u=u(x)} + C$$

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(4) Substitute for $F(u(x))$:

$$\int f(u(x)) \cdot u'(x)\,dx \;=\; F(u(x)) \;=\; \int f(u)\,du\,\Big|_{u=u(x)} + C$$

This is "u-substitution" in final form.

Integration by parts

Videos

Videos:

- Integration by Parts 1: $\int e^x\,dx$ and $\int \ln x\,dx$
- Integration by Parts 2: $\int \tan^{-1} x\,dx$ and $\int x \sec x\,dx$
- Integration by Parts 3: Definite integrals
- Example: $\int e^{3x} \cos 4x\,dx$, two methods:
  - Double IBP
  - Fast Solution
- Integration by Parts 6: $\int \sec^5 x\,dx$

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v:

$$\int u \cdot v'\,dx$$

Then the rule says we may convert the integral like this:

$$\int u \cdot v'\,dx \quad \ggg \quad u \cdot v - \int u' \cdot v\,dx$$

This technique comes from the **product rule for derivatives**:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u \cdot v = \int u' \cdot v\,dx + \int u \cdot v'\,dx$$

and the IBP rule follows by algebra.

Extra - Full explanation of integration by parts

(1) Product rule for derivatives:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

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(2) Integrate both sides:

$$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \gg\gg \quad \int (u \cdot v)'\, dx = \int u' \cdot v\, dx + \int u \cdot v'\, dx$$

$$\gg\gg \quad u \cdot v = \int u' \cdot v\, dx + \int u \cdot v'\, dx \qquad \text{(FTC)}$$

$$\gg\gg \quad \int u \cdot v'\, dx = u \cdot v - \int u' \cdot v \qquad \text{(Rearran}$$

Definite IBP >

Definite version of FTC:

$$\int_a^b (u \cdot v)'\, dx = u \cdot v \Big|_a^b$$

$$\gg\gg \quad \int_a^b u \cdot v'\, dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v?

Here is a trick: the acronym "LIATE" spells out the order of choices – to the left for u and to the right for v:

$$\text{LIATE :}$$

$$u \leftarrow \text{Logarithmic – Inverse\_trig – Algebraic – Trig – Exponential} \rightarrow v$$

05 Illustration

Example - A and T factors

Compute the integral: $\int x \cos x\, dx$

Solution

(1) Choose u and v':

Set $u(x) = x$ because x *simplifies* when differentiated.

(By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)

Remaining factor must be $v' = \cos x$.

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(2) Compute u' and v:

$$u = x \quad \gg\gg \quad u' = 1$$

$$v' = \cos x \quad \gg\gg \quad v = \sin x$$

Key chart:

$$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{ccc} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$

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(3) Evaluate IBP formula:

$$\int u'v\,dx \;=\; uv - \int uv'\,dx$$

$$\gg\gg \quad \int x\cos x\,dx \;=\; x\sin x - \int 1\cdot\sin x\,dx$$

$$\gg\gg \quad x\sin x + \cos x + C$$

Why IBP?

The *rationale* of IBP is that $\int 1 \cdot \sin x\,dx$ is easier to compute than $\int x\cos x\,dx$.

Exercise - Hidden A

Compute the integral:

$$\int \ln x\,dx$$

Solution

(1) Choose $u = \ln x$:

Because Log is farthest left in LIATE.

Therefore we must choose $v'(x) = 1$.

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(2) Compute u' and v:

We have $u' = \frac{1}{x}$ and $v = x$. Obtain chart:

$$\begin{array}{c|c} u = \ln x & v' = 1 \\ \hline u' = 1/x & v = x \end{array} \quad \begin{array}{ccc} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$

---

(3) Evaluate IBP formula:

$$\int u'v\,dx \;=\; uv - \int uv'\,dx$$

$$\gg\gg \quad \int \ln x \cdot 1\,dx \;=\; x\ln x - \int \frac{1}{x}\cdot x\,dx$$

Perform new integration:

$$-\int \frac{1}{x}\cdot x\,dx \quad \gg\gg \quad -\int 1\,dx \quad \gg\gg \quad -x + C$$

Final answer is: $x \ln x - x + C$