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1) $\int \sin x \cos^7 x \, dx$ 2) $\int \sin^3 x \, dx$ 3) $\int \sin^3 x \cos^2 x \, dx$

4) $\int \cos^2 x \sin^5 x \, dx$ 5) $\int \tan^5 x \sec^3 x \, dx$

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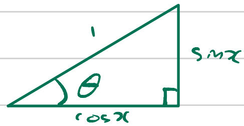
§ Trig Power Products

Reminder:



- $\sin^2 x + \cos^2 x = 1$
  "Pythagorean Identity"

- $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
- $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

$\Big\}$ pwr to freq. conversion

Three techniques ("moves"):

1. "Swap even bunch"
2. "u-sub for power-one"
3. "Power-to-frequency conversion"

Example

1. "Swap even bunch" $\sin^2 x \longrightarrow 1 - \cos^2 x$
   $\cos^2 x \longrightarrow 1 - \sin^2 x$

$$\sin^5 x \cos^8 x \rightsquigarrow \sin x \sin^4 x \cos^8 x$$
$$\rightsquigarrow \sin x (1 - \cos^2 x)^2 \cos^8 x$$
$$\rightsquigarrow (\cos^{12} x - 2\cos^{10} x + \cos^8 x) \sin x$$

Good! all but one are cosine...

"even bunch" = biggest even subfactor of odd power"

2. u-sub for power-one

$$\int \sin x \cos^8 x \, dx$$ ah... power one ($\sin^1 x$)

set $u = \cos x$ $du = -\sin x \, dx$

$$-\int u^8 \, du = -\frac{u^9}{9} + C = \boxed{-\frac{\cos^9 x}{9} + C}$$

So in above:

$$\int \sin x (1-\cos^2 x)^2 \cos^8 x \, dx \rightsquigarrow \int -(1-u^2)^2 u^8 \, du$$

$$\int (\cos^{12} x - 2\cos^{10} x + \cos^8 x) \sin x \, dx \rightsquigarrow -\int u^{12} - 2u^{10} + u^8 \, du$$

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1) $\int \sin x \cos^7 x \, dx$

2) $\int \sin^3 x \, dx$

$\sin^2 x \sin x \, dx$

1) u-sub directly:

$$\rightsquigarrow -\int u^7 du = -\frac{u^8}{8} + C$$

$$= -\frac{\cos^8 x}{8} + C$$

-OR-

$\cos^7 x = (1-\sin^2 x)^3 \cos x$

$$\rightsquigarrow \int \sin x (1-\sin^2 x)^3 \cos x \, dx$$

$$\rightsquigarrow \int u(1-u^2)^3 du = \int u^7 - 3u^5 - 3u^3 + u \, du$$

$$\rightsquigarrow \frac{u^8}{8} - \frac{3u^6}{6} - 3\frac{u^4}{4} + \frac{u^2}{2} + C \rightsquigarrow \frac{\sin^8 x}{8} - \frac{3\sin^6 x}{6} - \frac{3\sin^4 x}{4} + \frac{\sin^2 x}{2} + C$$

2) "even bunch" = $\sin^2 x$

$(1-\cos^2 x)$ Then

$$\rightsquigarrow -\int (1-u^2) \, du = \frac{u^3}{3} - u + C$$

$$= \boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

4)

$$\int \cos^2 x \sin^5 x \, dx$$

even bunch $= \sin^4 x$

So: $\sin^4 x = (\sin^2 x)^2 \rightsquigarrow (1-\cos^2 x)^2$

$$\rightsquigarrow \int \cos^2 x (1-\cos^2 x)^2 \sin x \, dx$$

$$\rightsquigarrow -\int u^2(1-u^2)^2 du = \int -u^6 + 2u^4 - u^2 du$$

$u = \cos x$, $du = -\sin x$

$$\rightsquigarrow -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C \rightsquigarrow \boxed{-\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$$

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3)

$$\int \sin^2 x \cos^2 x \, dx$$

even powers only (sine/cosine)

$\rightsquigarrow$ use pwr-to-freq.

$$\rightsquigarrow \int (1-\cos^2 x)\cos^2 x \, dx$$

$$= \int -\cos^4 x + \cos^2 x \, dx$$

$(\cos^2)^2$

$$\rightsquigarrow \int -\left(\frac{1}{2}(1+\cos 2x)\right)^2 + \frac{1}{2}(1+\cos 2x) \, dx$$

$$\int -\frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) + \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$$

$$-\frac{1}{4}\int \cos^2 2x \, dx + \underbrace{\int +\frac{1}{4} \cancel{- \frac{1}{2}\cos 2x + \frac{1}{2}\cos 2x} \, dx}_{= \frac{1}{4}x + C}$$

$$-\frac{1}{4}\int \frac{1}{2}(1+\cos(2 \cdot 2x)) \, dx$$

$$= \int -\frac{1}{8} - \frac{1}{8}\cos(4x) \, dx$$

$$= -\frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

$$\boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}$$

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Review: $\int \tan x\,dx \qquad \int \sec x\,dx$ $\qquad \tan^{-1}x \xrightarrow{d} \frac{1}{1+x^2}$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \frac{\sin x}{\cos x}dx \overset{u=\cos x}{\rightsquigarrow} -\int \frac{1}{u}\,du = -\ln|u| + C$$
$$= -\ln|\cos x| + C$$
$$= \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\hookrightarrow \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \tan x\sec x}{\tan x + \sec x}dx$$

Recall $\tan x \xrightarrow{d} \sec^2 x$

$\sec x \xrightarrow{d} \sec x\tan x$

$$\rightsquigarrow \int \frac{du}{u} = \ln|u| + C \quad \text{w/ } u = \tan x + \sec x \ \checkmark$$

5) $\int \tan^5 x \sec^3 x \, dx = ??$

Use:

- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

$$d(\overbrace{\tan x}^{u}) = \overbrace{\sec^2 x \, dx}^{du}$$

$$d(\underbrace{\sec x}_{u}) = \underbrace{\sec x \tan x \, dx}_{du}$$

$$\int \tan^m x \sec^n x \, dx$$

Eg:

$\int \tan^2 x \sec^2 x \, dx$ ✓

$\int u^2 du$

$\int \tan x \sec^2 x \, dx$ ✓

$u = \tan$

$du = \sec^2 dx$

$\int u \, du$

$\int \tan^2 x \sec x \, dx$ ✗

$\int \tan^4 x \, dx$ ✓

$\tan^2(\sec^2 x - 1)$

$-\int \tan^2 x \, dx + \int u^2 du$

$\int (\sec^2 x - 1) dx$

$\int \sec^2 x \, dx = \tan x + c$

$\int \tan^3 x \, dx$ ✓

$\int \tan x (\sec^2 x - 1) dx$

$= -\int \tan x \, dx + \int \underbrace{\tan x (\sec^2 x \, dx)}_{= \int u \, du}$

$\int \sec^3 x \, dx$ ✗

$(\tan^2 + 1) \sec \, dx$

$\int \tan^2 \sec \, dx + \int \sec \, dx$

Can do:

"even secants"

or "odd tangents" (w/ catch)

| $n > 0$ even ✓
| m odd & $n \geq 1$ ✓

$n = 0$, m odd ✗

m even, n odd ✗

5) $\int \tan^5 x \sec^3 x \, dx$

Try: $du = \sec^2 x \, dx \rightsquigarrow u = \tan x$

$\int \tan^5 x \sec^3 x \, dx \rightsquigarrow \int \tan^5 x \sec x \, du$

stuck!

Try: $du = \sec x \tan x \, dx$, $u = \sec x$

$$\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \, du$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \, du$$

$$= \int (u^2-1)^2 u^2 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^2 \, du = \int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

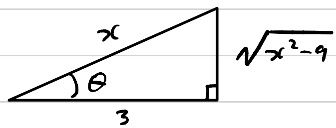
$$= \boxed{\frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C}$$

Trig Substitution

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx \qquad\qquad \int \frac{dx}{\sqrt{x^2-6x+11}}$$

- simplify quadratic polynomials
- eliminate square root of quadratic

Example: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$ Notice $\sqrt{x^2-9}$ Pythagorean:



From triangle:

$\sqrt{x^2-9} = 3\tan\theta$

$x = 3\sec\theta \rightsquigarrow dx = 3\sec\theta\tan\theta\,d\theta$

$$\rightsquigarrow \int \frac{1}{(3\sec\theta)^2\,\cancel{3\tan\theta}}\,\cancel{3}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$

$$\rightsquigarrow \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C = \boxed{\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C}$$

Methodical Method:

i.e. $a^2 - a^2\sin^2\theta = a^2\cos^2\theta$

$\sqrt{a^2-x^2}$: use $x = a\sin\theta$ from $1-\sin^2\theta = \cos^2\theta$

$\sqrt{a^2+x^2}$: use $x = a\tan\theta$ from $1+\tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$: use $x = a\sec\theta$ from $\sec^2\theta - 1 = \tan^2\theta$

E.g. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$: use $x = 3\sec\theta \rightsquigarrow dx = 3\sec\theta\tan\theta\,d\theta$

$\displaystyle\int \frac{1}{(3\sec\theta)^2\,3\tan\theta}\,3\sec\theta\tan\theta\,d\theta$ $\qquad \sqrt{x^2-9} = \sqrt{9\sec^2\theta-9} = 3\sqrt{\tan^2\theta}$

$= 3\tan\theta$

$\hookrightarrow \frac{1}{9}\sin\theta + C = \frac{1}{9}\sin\left(\sec^{-1}\left(\frac{x}{3}\right)\right) + C$

must simplify... use a triangle...