# Partial Fractions

"Rational functions"

$$\frac{3x^3-5x^2-6x+20}{x^4-3x^3+4x} = -\frac{2}{x+1} + \frac{2}{(x-2)^2} + \frac{5}{x}$$

$\int \nearrow -2\ln|x+1|$ $\quad \int \nearrow -\frac{2}{x-2}$ $\quad \int \nearrow 5\ln|x|$

- Denominators: factor $\rightsquigarrow$ Linear, $(\text{linear})^n$
  $\rightsquigarrow$ irreducible quadratic, $(\text{irred quad})^n$
  $b^2 < 4ac$
- Numerator: Lower degree

E.g. <u>valid</u>: $\frac{1}{x^2+1}$, $\frac{2x+1}{x^2+5}$, $\frac{7}{5x-8}$, $\frac{1}{x}$, $\frac{1}{x^3}$

<u>invalid</u>: $\frac{x}{x-1}$, $\frac{x^3+2}{x^2+1}$, $\frac{1}{x^2-1}$, $\frac{1}{x(x-1)}$

$\hookrightarrow (x-1)(x+1)$

<u>valid</u>: $\frac{x}{x^2+1}$, $\frac{x^3+2x+1}{(x^2+2)^2}$

## PFD Procedure

1. Check denom is higher degree
   $\rightsquigarrow$ Else: Long division
2. Factor denom <u>completely</u> (even irrational roots)
3. Write generic PFD w/ constants
   Higher pwrs/Repeated factors $\rightsquigarrow$ incrementing powers
4. Solve for those constants

**Example:** $\displaystyle\int \frac{3x-9}{x^3+3x^2-4}\,dx$

1. Yes: num deg $= 1 <$ denom deg $= 3$ ✓

2. Factor denom completely:

Rational Roots Theorem: guess $\pm 1, \pm 2, \pm 4$

$x = +1$ is a root $\rightsquigarrow$ $(x-1)$ is a factor $\xrightarrow{\text{factor again}}$ $(x+2)^2$

$$
\begin{array}{r|l}
 & x^2 + 4x + 4 \\
\hline
x-1 & x^3 + 3x^2 + 0x - 4 \\
 & \underline{x^3 - x^2} \\
 & \quad\;\; 4x^2 + 0x \\
 & \quad\;\; \underline{4x^2 - 4x} \\
 & \qquad\qquad 4x - 4 \\
 & \qquad\qquad \underline{4x - 4} \\
 & \qquad\qquad\quad\; 0
\end{array}
$$

Thus: $x^3 + 3x^2 - 4$
$= (x-1)(x+2)^2$

e.g. $\frac{1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$

3. Generic PFD:

$$\frac{3x-9}{x^3+3x^2-4} = \frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

4. Solve for $A$, $B$, $C$:

Cross multiply:

$$3x - 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

(true all $x$ values)

Plug in "inspired" values:

@ $x=-2$: $3(-2)-9 = 0 + 0 + C(-3)$

$\rightsquigarrow -15 = -3C \rightsquigarrow C = 5$

@ $x=+1$: $3(1)-9 = A(3)^2 + 0 + 0$

$\rightsquigarrow -6 = 9A \rightsquigarrow A = -2/3$

@ $x=3$: $0 = -\frac{2}{3}(5)^2 + B(2)(5) + 5(2)$

$\rightsquigarrow B = 2/3$

Therefore:

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{-2/3}{x-1} + \frac{2/3}{x+2} + \frac{5}{(x+2)^2}$$

$\rightsquigarrow$

$$\int \frac{3x-9}{(x-1)(x+2)^2}\,dx = \int \frac{-2/3}{x-1}\,dx + \int \frac{2/3}{x+2}\,dx + \int \frac{5}{(x+2)^2}\,dx$$

$$= \boxed{-\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| - \frac{5}{x+2} + C}$$

# Integration of PFD:

$$\int \frac{A}{x-a} \quad \text{ln}$$

$$\int \frac{B}{(x-a)^2} \quad \text{pwr rule}$$

$$\cdots$$

$$\int \frac{C}{(x-a)^n} \quad \text{pwr rule}$$

$x = h\tan\theta$
$dx = h\sec^2\theta\, d\theta$
$x^2+h^2 = h^2\sec^2\theta$

$$\rightsquigarrow \int \frac{A h \sec^2\theta\, d\theta}{h^2\sec^2\theta}$$

$$= \frac{A}{h}\int d\theta = \frac{A}{h}\theta + C$$

$$= \frac{A}{h}\tan^{-1}\left(\frac{x}{h}\right) + C$$

$$\int \frac{A}{x^2+h^2}$$

memorize!

$$\frac{A}{h}\tan^{-1}\left(\frac{x}{h}\right) + C$$

$$\int \frac{Ax+B}{x^2+h^2}$$

split up

$$\int \frac{Ax}{x^2+h^2}\,dx + \int \frac{B}{x^2+h^2}\,dx$$

$u = x^2+h^2$
$du = 2x\,dx$

$$\frac{1}{2}\int \frac{A\,2x\,dx}{x^2+h^2} = \frac{1}{2}\int \frac{A\,du}{u}$$

$$\frac{A}{2}\ln|x^2+h^2| + C$$

same as left

$$\frac{B}{h}\tan^{-1}\left(\frac{x}{h}\right) + C$$

Example: $\displaystyle\int \frac{x^3+1}{(x^2+4)^2}\,dx$

1. Yes deg num $= 3 < 4 =$ deg denom ✓

2. Factored already.

3. Generic PFD:

- irred. quadratic
- higher power

$$\frac{x^3+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

Cross mult.: $x^3+1 = (Ax+B)(x^2+4) + (Cx+D)$

$$\rightsquigarrow \quad x^3+1 = Ax^3 + Bx^2 + (4A+C)x + (4B+D)$$

$$\rightsquigarrow \quad A = 1, \quad B = 0$$
$$C = -4, \quad D = 1$$

$$\int \frac{x^3+1}{(x^2+4)^2} = \frac{x}{x^2+4} + \frac{-4x+1}{(x^2+4)^2} \, dx$$

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + C$$

$$\int \frac{-4x}{(x^2+4)^2}\,dx = -2\int \frac{du}{u^2} = \frac{2}{u} + C = \frac{2}{x^2+4} + C$$

$x = 2\tan\theta$

$$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\frac{x}{x^2+4} + C$$

Example:

$$\int \frac{3x^2+2}{(x^2+1)\,x(x-1)}\,dx$$

1. Yes, $2 < 4$ (num lower deg)
2. Already factored.

3. Write generic PFD:

$$\frac{3x^2+2}{(x^2+1)\,x(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x} + \frac{D}{x-1}$$

4. Solve:

Cross multiply:

$$3x^2+2 = (Ax+B)x(x-1) + C(x^2+1)(x-1) + Dx(x^2+1)$$

@ $x=0$: $2 = 0 + C(1)(-1) + 0 \rightsquigarrow C = -2$

@ $x=1$: $5 = 0 + 0 + D(1)(2) \rightsquigarrow D = 5/2$

$\frac{d}{dx}$: $6x = Ax(x-1) + \overset{(Ax+B)(2x-1)}{\cancel{(Ax+B)(x-1) + (Ax+B)x}}$
$+ \underset{C(3x^2-2x+1)}{\cancel{C(2x)(x-1) + C(x^2+1)}} + \underset{D(3x^2+1)}{\cancel{D(x^2+1) + Dx(2x)}}$

@ $x=2$: $14 = (2A+B)2(1) + \overset{-2}{\cancel{C}}(5)(1) + \overset{5/2}{\cancel{D}}(2)(5)$

$14 = 4A + 2B + 15$

$-1 = 4A + 2B$

@ $x=-1$: $5 = (-A+B)(-1)(-2) - 2(2)(-2) + \frac{5}{2}(-1)(2)$

$5 = -2A + 2B + 8 - 5$

$2 = -2A + 2B \qquad 1 = -A + B \qquad A = -1 + B$

$-1 = 4(-1+B) + 2B = +4B + 2B - 4 = \overset{6}{\cancel{2}}B - 4$

$3 = \underset{6}{\cancel{2}}B, \quad B = \underset{+1/2}{\cancel{-3/2}}, \quad A = -1/2$

$$\int \frac{3x^2+2}{(x^2+1)\,x\,(x-1)} = \frac{\overset{-\frac{1}{2}x+\frac{1}{2}}{\cancel{\frac{1}{2}x - \frac{3}{2}}}}{x^2+1} + \frac{-2}{x} + \frac{5/2}{x-1}$$

$$\cancel{\frac{1}{2}\left(\frac{x}{x^2+1}\right) - \frac{3}{2}\cdot\frac{1}{x^2+1}} \qquad -\frac{1}{4}\cdot\frac{2x}{x^2+1} + \frac{1/2}{x^2+1}$$

$$\leadsto \boxed{-\frac{1}{4}\ln|x^2+1| + \frac{\cancel{3}\,1}{2}\tan^{-1}(x) - 2\ln|x| + \frac{5}{2}\ln|x-1| + C}$$