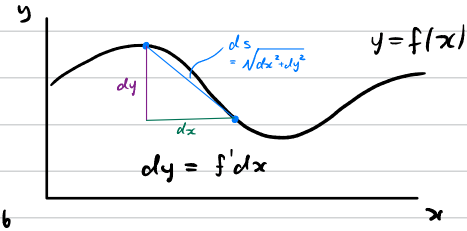
Arc Length

$$L = \int_a^b \sqrt{1+(f')^2}\, dx$$



$$L = \int_a^b ds = \int_a^b \sqrt{dx^2+dy^2} = \int_a^b \sqrt{dx^2+(f'dx)^2} = \int_a^b \sqrt{1+(f')^2}\, dx$$

Example: Find arc length of $y = \ln x$, $1 \le x \le \sqrt{3}$.

$\rightsquigarrow f' = \frac{1}{x}$

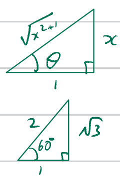
Solution: $L = \int_1^{\sqrt{3}} \sqrt{1+(1/x)^2}\, dx$

$1 + \left(\frac{1}{x}\right)^2 = \frac{x^2+1}{x^2}$

$\sqrt{\cdot} \rightsquigarrow \frac{\sqrt{x^2+1}}{x}$

$$\rightsquigarrow \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\, dx$$

$x = \tan\theta$
$dx = \sec^2\theta\, d\theta$
$1 \rightsquigarrow \theta = \pi/4$
$\sqrt{3} \rightsquigarrow \theta = \pi/3$



$$\rightsquigarrow \int_{\pi/4}^{\pi/3} \frac{\sec^3\theta}{\tan\theta}\, d\theta \rightsquigarrow \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta$$

$$\rightsquigarrow \int_{\pi/4}^{\pi/3} \csc\theta + \sec\theta\tan\theta\, d\theta \rightsquigarrow \ln|\csc\theta - \cot\theta| + \sec\theta \Big|_{\pi/4}^{\pi/3}$$

$$\rightsquigarrow \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| + 2 - \left(\ln|\sqrt{2}-1| + \sqrt{2}\right)$$

$$= \boxed{\ln\left|\frac{\sqrt{3}(\sqrt{2}+1)}{3}\right| + 2 - \sqrt{2}}$$

Example: Arc length of $y = \frac{x^4}{8} + \frac{1}{4x^2}$, $x \in [1,2]$

Solution:

$$f' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$$

$$(f')^2 = \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}$$

$$1+(f')^2 = \frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}$$ insight $\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$

$$\sqrt{1+(f')^2} = \frac{1}{2}x^3 + \frac{1}{2}x^{-3}$$

$$\int_1^2 \sqrt{1+(f')^2}\,dx = \int_1^2 \frac{1}{2}x^3 + \frac{1}{2}x^{-3}\,dx = \boxed{\frac{33}{16}}$$

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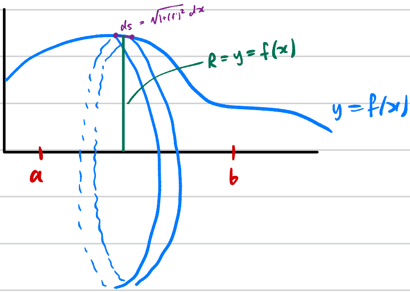
Surface Area of Revolutions



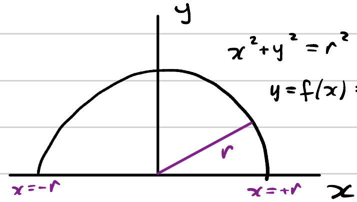
$$S = \int_a^b 2\pi R\, ds \xrightarrow[\text{revolve around } x\text{-axis}]{y=f(x)} \int_a^b 2\pi f(x)\sqrt{1+(f')^2}\,dx$$

Example: Calculate surface area of a sphere.

Solution: Expect $4\pi r^2$.

$x^2+y^2=r^2$

$y=f(x)=\sqrt{r^2-x^2}$



$$S = \int_a^b 2\pi R\, ds \rightsquigarrow \int_{-r}^{+r} 2\pi\sqrt{r^2-x^2}\,\sqrt{1+(f')^2}\,dx$$

$$f = \sqrt{r^2-x^2}$$

$$f' = +\tfrac{1}{2}(r^2-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$$

$$(f')^2 = \frac{x^2}{r^2-x^2}$$

$$1+(f')^2 = \frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2} = \frac{r^2}{r^2-x^2}$$

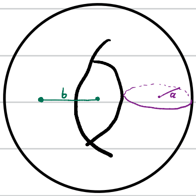
$$\rightsquigarrow \frac{r}{\sqrt{r^2-x^2}}$$

$$\rightsquigarrow S = \int_{-r}^{+r} 2\pi r\, dx \rightsquigarrow 2\pi r x\Big|_{-r}^{+r} = 2\pi r r - 2\pi r(-r)$$

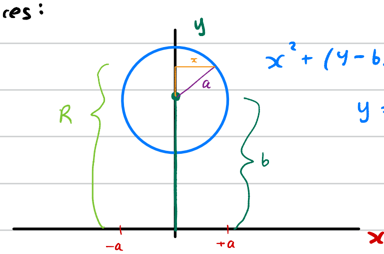
$$= \boxed{4\pi r^2}$$

Example: Surface area of a torus = "donut".

Solution:



circumferences:
$2\pi a$
$2\pi b$



$$x^2 + (y-b)^2 = a^2$$

$$y = b \pm \sqrt{a^2 - x^2}$$

$$S = \int_{-a}^{+a} 2\pi\left(b + \cancel{\sqrt{a^2-x^2}}\right)\sqrt{1+(f')^2}\,dx + \int_{-a}^{+a} 2\pi\left(b - \cancel{\sqrt{a^2-x^2}}\right)\sqrt{1+(f')^2}\,dx$$

$$\int_{-a}^{+a} 2\pi b\sqrt{1+f'^2}\,dx + \int_{-a}^{+a} 2\pi\sqrt{a^2-x^2}\sqrt{1+f'^2}\,dx$$

$$+ \int_{-a}^{+a} 2\pi b\sqrt{1+f'^2}\,dx - \int_{-a}^{+a} 2\pi\sqrt{a^2-x^2}\sqrt{1+f'^2}\,dx$$

$$f = b \pm \sqrt{a^2-x^2}$$

$$f' = \frac{-2x}{2\sqrt{a^2-x^2}}$$

$$1+(f')^2 = \frac{a^2}{a^2-x^2}$$

$$\rightsquigarrow \int_{-a}^{+a} 4\pi b\sqrt{1+(f')^2}\,dx \rightsquigarrow \int_{-a}^{+a} 4\pi b \frac{a}{\sqrt{a^2-x^2}}\,dx$$

$$x = a\sin\theta$$
$$dx = a\cos\theta\, d\theta$$

$$\rightsquigarrow \int_{-\pi/2}^{+\pi/2} \frac{4\pi a b\, \cancel{a\cos\theta}\, d\theta}{\cancel{a\cos\theta}} \rightsquigarrow 4\pi a b\,\theta \Big|_{-\pi/2}^{+\pi/2}$$

$$\rightsquigarrow 4\pi^2 a b$$

$$\rightsquigarrow \boxed{(2\pi a)(2\pi b)}$$