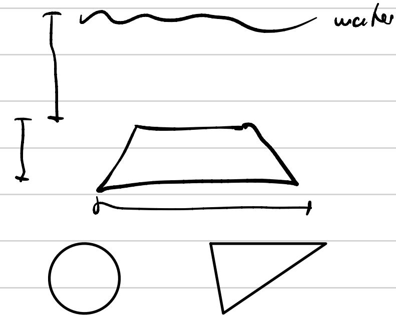
Hydrostatic Force



Force = Pressure × Area

Pressure = Depth × ρg

slugs/ft^3

- ρ = mass density e.g. $1000 \frac{kg}{m^3}$ H_2O
- g = gravity constant = $9.8 \frac{m}{s^2}$

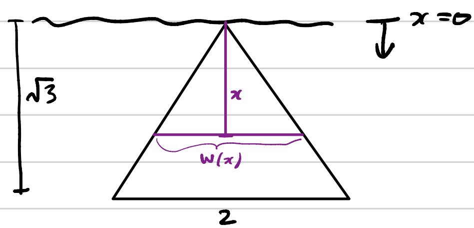
$$F = \int_a^b \rho g\, h(x)\, w(x)\, dx$$

a, b = x @ top / bottom of shape

$h(x)$ = depth under water line @ x

$w(x)$ = width of shape @ x

Example:



1. Specify coordinate.
2. Depth $h(x) = x$.
3. Find $w(x)$:

Similar Triangles: $\frac{w(x)}{x} = \frac{2}{\sqrt{3}}$

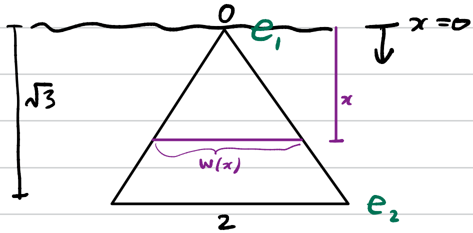
$\rightsquigarrow w(x) = \frac{2}{\sqrt{3}} x$.

4. Set up integral:

$$F = \int_0^{\sqrt{3}} \rho g x \left(\frac{2}{\sqrt{3}} x\right) dx \rightsquigarrow \rho g \frac{2}{\sqrt{3}} \int_0^{\sqrt{3}} x^2\, dx = \boxed{17640}$$

Method 2: "QLIF"

Quick Linear Interpolation Function



$$w(x) = e_1 + \frac{e_2 - e_1}{d}(x-a)$$

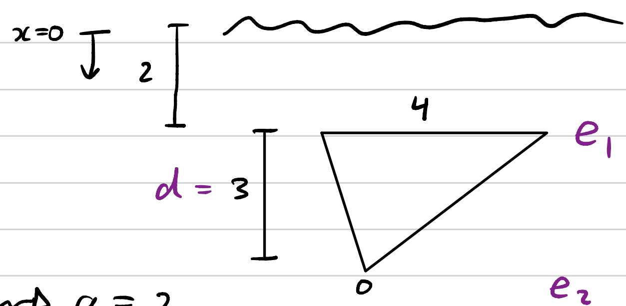
choose a so $(x-a)=0$ @ $x = e_1$

So here: $w(x) = 0 + \frac{2-0}{\sqrt{3}}(x-a)$

$(x+a)=0$ @ e_1 $x=0$ $\rightsquigarrow$ $a=0$

$\rightsquigarrow$ $w(x) = \frac{2}{\sqrt{3}}x$

Example:



$h(x) = x$

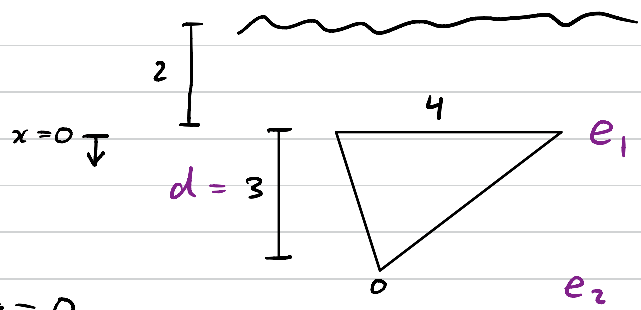
$$w(x) = 4 + \frac{0-4}{3}(x-a)$$

$(x-a)=0$ @ $x=2$ $\rightsquigarrow$ $a=2$

$\rightsquigarrow$ $w(x) = 4 - \frac{4}{3}(x-2)$ $\rightsquigarrow$ $\frac{20}{3} - \frac{4}{3}x$

$$\boxed{F = \int_{x=2}^{5} \rho g x \left(\frac{20}{3} - \frac{4}{3}x\right) dx}$$

Example:



$h(x) = x + 2$

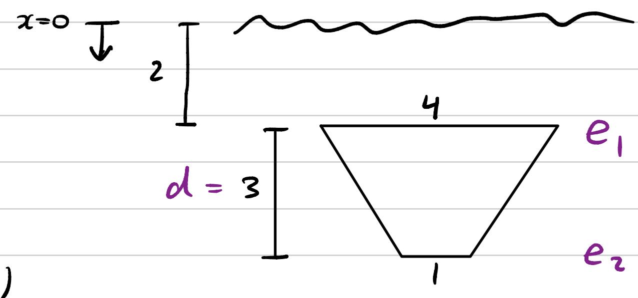
$w(x) = 4 + \frac{0-4}{3}(x-a)$

$(x-a) = 0$ @ $x = 0$ and $a = 0$

and $w = 4 - \frac{4}{3}x$

$$\boxed{F = \int_0^3 \rho g (x+2)\left(4 - \frac{4}{3}x\right) dx}$$

Example:



1. $x=0$ @ water line.
2. $h(x) = x$
3. $W(x)$:

$w(x) = 4 + \frac{1-4}{3}(x-a)$

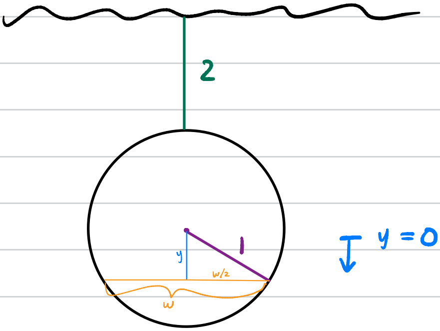
$(x-a) = 0$ at $x = 2$ (at e_1) and $a = 2$

$$\boxed{F = \int_2^5 \rho g x \left(4 - \frac{3}{3}(x-2)\right) dx}$$

Example:

$$h(y) = y + 3$$

$$w(y) = 2\sqrt{1-y^2}$$



$$\boxed{F = \int_{y=-1}^{+1} \rho g\,(y+3)\, 2\sqrt{1-y^2}\, dy}$$

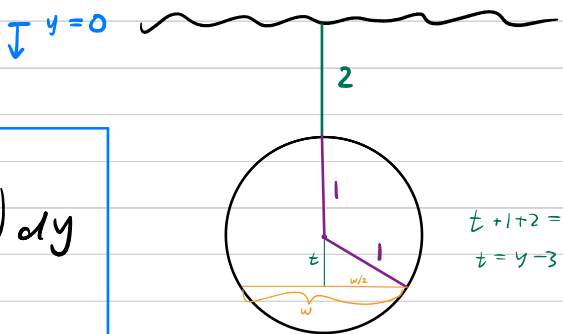
$$y^2 + \left(\frac{w}{2}\right)^2 = 1^2$$

$$\rightsquigarrow \frac{w}{2} = \sqrt{1-y^2}$$

$$w = 2\sqrt{1-y^2}$$

Try at water line:

$$h(y) = y$$



$t + 1 + 2 = y$

$t = y - 3$

$$\boxed{F = \int_{2}^{4} \rho g\, y \left(2\sqrt{1-(y-3)^2}\right) dy}$$

$$t^2 + \left(\frac{w}{2}\right)^2 = 1^2$$

$$\rightsquigarrow \frac{w}{2} = \sqrt{1-t^2}$$

$$w = 2\sqrt{1-t^2}$$

$$w = 2\sqrt{1-(y-3)^2}$$

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Total Work Performed

– layer by layer –

Work = Force × Distance

$$W = \int_a^b F(x)\,dx \qquad \text{-OR-} \qquad W = \int_a^b h(x)\,dF$$

(Later, leaking bucket)

"sequential lifting" "parallel lifting"

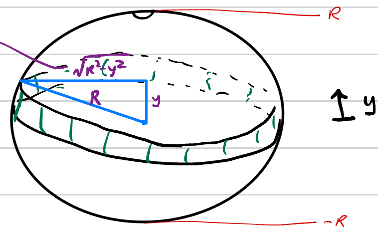
Lifting layers: $dW = \text{height raised} \times \rho g \times A(x)\,dx$

$$W = \int dW = \int_a^b \rho g\, h(x)\, A(x)\,dx$$

- $A(x)$ = area of layer
- $h(x)$ = height of layer
- ρ = mass density $= 1000\ ^{kg}/_{m^3}$ for H_2O
- g = gravity constant $= 9.8\ \frac{m}{s^2}$

Example:

$r = \sqrt{R^2-y^2}$



$$W = \int_a^b \rho g h A\,dy$$

$A(y) = \pi r^2 = \pi\left(\sqrt{R^2-y^2}\right)^2 = \pi(R^2-y^2)$

$h(y)$ = dist. y up to R $= R-y$

a = bot. $= -R$, b = top $= +R$

$$\rightsquigarrow \boxed{W = \int_{-R}^{+R} \rho g (R-y)\pi(R^2-y^2)\,dy}$$

plug $\rho = 1000$, $g = 9.8$, $R = 5m$ $\rightsquigarrow$ $\boxed{2.6\times10^7\ J}$

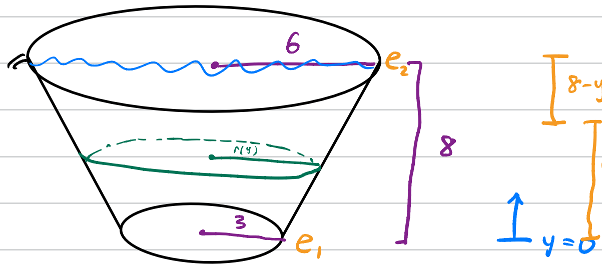
Q: How to change if only $\frac{3}{4}$ full? (Measured by height of H_2O)

$$\rightsquigarrow \quad W = \int_{-R}^{+R/2} \rho g (R-y)\pi(R^2-y^2)\,dy$$

Example:

$$W = \int_a^b \rho g h A \, dy$$

1. ALWAYS clearly indicate your coordinate!



2. $h(y) = 8 - y$

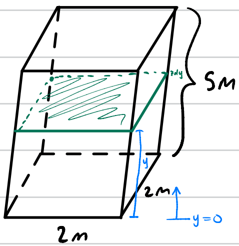
3. $a = 0, \quad b = 8$

4. $A(y) = \pi r(y)^2, \quad r(y) = 3 + \overset{\text{"QLIF"}}{\frac{6-3}{8}}(y-a)$ $\quad (y-a) = 0$ @ $y=0 \rightsquigarrow a = 0$

$$= 3 + \frac{3}{8}y$$

$$\rightsquigarrow \boxed{W = \int_0^8 \rho g (8-y) \pi \left(3 + \frac{3}{8}y\right)^2 dy}$$

Example: Work to build a big block building



$$W = \int_a^b dW$$ work to place single layer at y

$$= \int_a^b \rho g h A\,dy$$

$A\,dy = dV$

$\rho A\,dy = dM$

$g\rho A\,dy = dF$

$h \times g\rho A\,dy = dW$

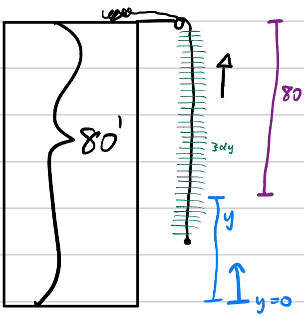
$h(y) = y$

$A(y) = 4\,m^2$

$a = 0,\ b = 5$

$$\rightsquigarrow W = \int_0^5 \rho g y (4)\,dy = \boxed{735\,kJ}$$

Example: $\circ\ \rho = 0.5\ ^{lb}/_{ft}$ Linear weight density



Q: Work to hoist chain.

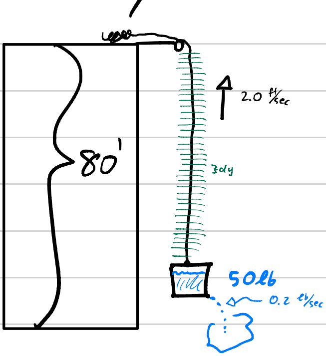
A: Vertical slices "links" dy

dF = weight of link $= 0.5\,dy$

$h(y) = 80 - y$ dist. y to top

$$W = \int_0^{80} (80-y)(0.5)\,dy = \boxed{1600\ \text{ft-lb}}$$

Example:



- Chain = weightless
- Bucket initial = 50 lb
- Leak rate = 0.2 lb/sec
- Lift rate = 2.0 ft/sec

Q: Work to hoist bucket.

A: $\dfrac{\text{leak per sec}}{\text{lift per sec}} = \text{leak per lift}$

$$\frac{0.2 \ \text{lb/sec}}{2.0 \ \text{ft/sec}} = 0.1 \ \text{lb/ft}$$

$$F(y) = 50 \text{ lb} - \left(0.1 \tfrac{\text{lb}}{\text{ft}}\right) y$$

$$W = \int_0^{80} dW = \int_0^{80} F(y)\,dy = \int_0^{80} (50 - 0.1y)\,dy = \boxed{3680 \text{ ft}\cdot\text{lb}}$$

vs. $W = \int h\,dF$ as in prev. problems (see first page)