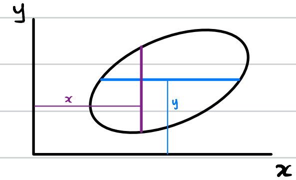
Moments & Center of Mass

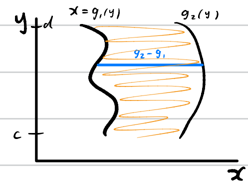
(x,y) $\bullet M$ $\quad M_x = M\cdot y$, $M_y = M\cdot x$

Given a "lamina" = plate w/ layers



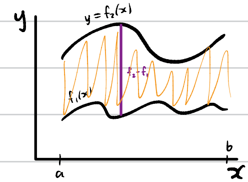
Moment to x-axis: $M_x = \int \rho y \, dA$

$$= \int_c^d \rho y (g_2 - g_1)\, dy$$



Moment to y-axis: $M_y = \int \rho x \, dA$

$$= \int_a^b \rho x (f_2 - f_1)\, dx$$



Center of Mass: CoM = the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}, \quad M = \text{total mass}$$

Why? $$\bar{x} = \frac{\int x \rho \, dA}{\int \rho \, dA} = \frac{\int x \, dM}{\int dM} =$$ weighted average of x values, weighted by strip mass dM

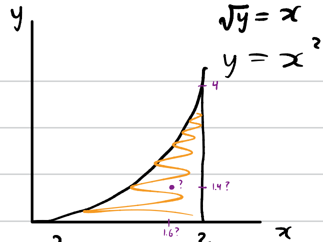
Weighted Average: numbers x_1, x_2, x_3

weights $c_1, c_2, c_3 \geq 0$

$$\text{wt'd avg} = \frac{c_1 x_1 + c_2 x_2 + c_3 x_3}{c_1 + c_2 + c_3} = \bar{x}$$

e.g. ◦ $c_1 = c_2 = c_3 = 1$ ⤳ reg. avg. $\bar{x} = \dfrac{x_1 + x_2 + x_3}{3}$

◦ $c_1 = 2$, $c_2 = c_3 = 1$ ⤳ double of x_1: $\bar{x} = \dfrac{x_1 + x_1 + x_2 + x_3}{4}$



Example: Find CoM of :

Solution: 1. Guess : $(\bar{x}, \bar{y}) = (1.6, 1.4)$

2. Find total mass $M = \int_0^2 \rho(f_2 - f_1)\,dx = \int_0^2 \rho(x^2 - 0)\,dx = \frac{8\rho}{3}$

3. Find $M_y = \int_0^2 \rho x (f_2 - f_1)\,dx = \int_0^2 \rho x (x^2 - 0)\,dx = \rho \frac{x^4}{4}\Big|_0^2 = 4\rho$

4. Find $M_x = \int_0^4 \rho y (g_2 - g_1)\,dy = \int_0^4 \rho y (2 - \sqrt{y})\,dy$

$$\rightsquigarrow \rho \int_0^4 2y - y^{3/2}\,dy \rightsquigarrow 2\rho \frac{y^2}{2} - \rho \frac{y^{5/2}}{5/2}\Big|_0^4 = \frac{16\rho}{5}$$

$M_x = \int_0^2 \frac{\rho}{2}\left((x^2)^2 - 0^2\right)dx$
$= \frac{\rho}{2}\int_0^2 x^4\,dx = \frac{\rho}{10}x^5\Big|_0^2$
$= \frac{16\rho}{5}$

5. Plug in for $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{M_y}{M} = \frac{4\rho}{8\rho/3} = \frac{3}{2} = 1.5$$

$$\bar{y} = \frac{M_x}{M} = \frac{16\rho/5}{8\rho/3} = \frac{6}{5} = 1.2$$

So CoM $= (\bar{x}, \bar{y}) = \boxed{(1.5,\ 1.2)}$

Downside: • for M_x, needed $f^{-1}(y) = g_2(y) = \sqrt{y}$
• finding f^{-1} can be hard / impossible

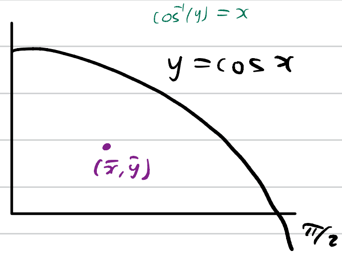
New technique:

$$M_x = \int_a^b \rho \tfrac{1}{2}(f_2^2 - f_1^2)\,dx, \quad M_y = \int_c^d \rho \tfrac{1}{2}(g_2^2 - g_1^2)\,dy$$

From above:

$$M_x = \int_c^d \rho(g_2 - g_1)\,dy, \quad M_y = \int_a^b \rho(f_2 - f_1)\,dx$$

Example:

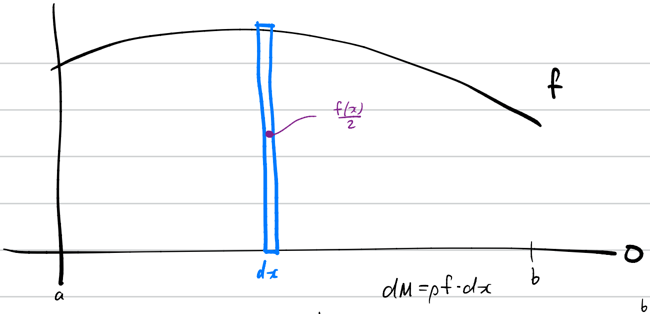


$$M = \int_0^{\pi/2} \rho \cos x\,dx = \rho$$

$$M_x = \int_0^{\pi/2} \rho \tfrac{1}{2}(\cos^2 x - 0^2)\,dx = \frac{\pi}{8}\rho$$

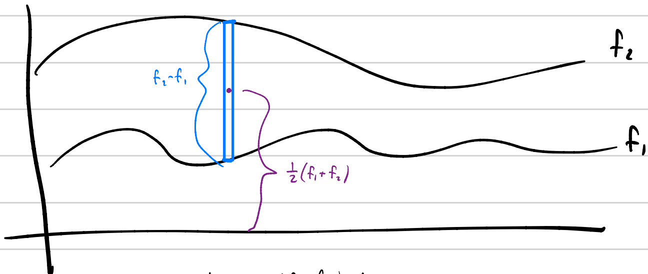
$$M_y = \int_0^{\pi/2} \rho x(\cos x - 0)\,dx = \rho\left(\frac{\pi}{2} - 1\right)$$

$$(\bar{x}, \bar{y}) = \left(\frac{\rho(\pi/2 - 1)}{\rho}, \frac{\pi\rho/8}{\rho}\right) = \boxed{\left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)}$$



$dM = \rho f \cdot dx$

$$\rightsquigarrow \int_a^b \left(\frac{f(x)}{2}\right)\left(\rho f(x)\, dx\right) \rightsquigarrow \int_a^b \frac{\rho}{2} f^2\, dx$$



$dm = (f_2 - f_1)\, dx$

$CoM = (x, \frac{1}{2}(f_1 + f_2))$

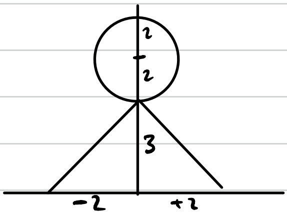
i.e. @ y-value $\frac{1}{2}(f_1 + f_2)$

$$\rightsquigarrow \int_a^b \rho \frac{1}{2}(f_1 + f_2)(f_2 - f_1)\, dx$$

$$\rightsquigarrow \int_a^b \frac{\rho}{2}(f_2^2 - f_1^2)\, dx \quad \checkmark$$

CoM principles : additivity, symmetry

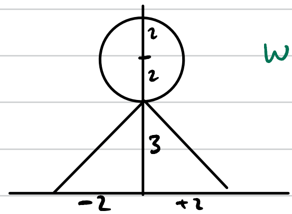
Additivity :



$$M_x^{tot} = M_x^{tri} + M_x^{circ}$$

$$M_y^{tot} = M_y^{tri} + M_y^{circ}$$

Symmetry :



$$w = e_1 + \frac{e_2 - e_1}{h}(y-a)$$

- CoM^{tot} on y-axis $\rightsquigarrow \bar{x}^{tot} = 0$
- CoM^{circ} @ $(0,5) = (\bar{x}^{circ}, \bar{y}^{circ})$

Example: Find CoM of above combo shape.

Solution: Have $\bar{x}^{tot} = 0$. Need $\bar{y}^{tot}$. Have $\bar{y}^{circ} = 5$.

1. $M_x^{circ} = m^{circ} \cdot \bar{y}^{circ}$ $\quad \left(\text{from } \bar{y}^{circ} = \dfrac{M_x^{circ}}{M^{circ}} \right)$
$$= (\rho \pi 2^2)(5) = 20\pi\rho$$

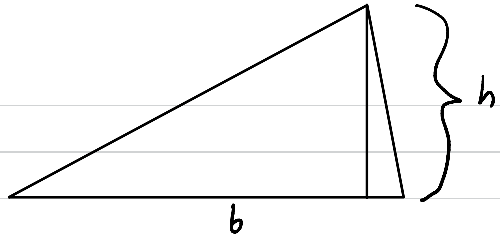
2. $$M_x^{tri} = \int_0^3 \rho y\, w(y)\, dy = \int_0^3 \rho y \left(4 + \frac{0-4}{3} y\right) dy = \int_0^3 \rho y \left(4 - \tfrac{4}{3} y\right) dy$$
$$= 6\rho$$

3. $$M_x^{tot} = M_x^{circ} + M_x^{tri} = \rho(20\pi + 6)$$

4. $$\bar{y}^{tot} = \frac{M_x^{tot}}{m^{tot}} = \frac{\rho(20\pi+6)}{4\pi\rho + 6\rho} = \frac{20\pi+6}{4\pi+6} \approx 3.71$$

Therefore : $\boxed{(\bar{x}, \bar{y}) = (0, 3.71)}$

Any triangle:



$$\bar{y} = \frac{M_x}{M}$$

$$M = \rho \frac{1}{2} b h$$

$$M_x = \int_0^h \rho y \, w(y) \, dy = \int_0^h \rho y \left(b + \frac{0-b}{h} y\right) dy$$

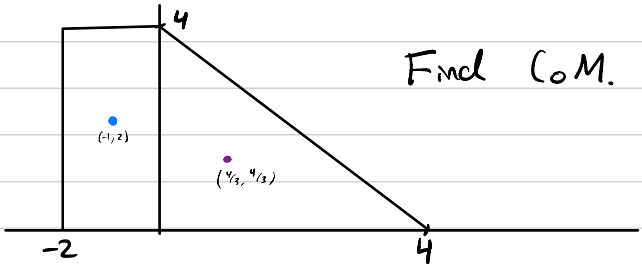
$$= \int_0^h \rho y \left(b - \frac{b}{h} y\right) dy = \frac{\rho b}{2} y^2 - \frac{\rho b}{3h} y^3 \Big|_0^h$$

$$= \frac{\rho b}{2} h^2 - \frac{\rho b}{3h} h^3$$

$$= \rho b h^2 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6} \rho b h^2$$

$$\hookrightarrow \bar{y} = \frac{\frac{1}{6} \rho b h^2}{\frac{1}{2} \rho b h} = \boxed{\frac{1}{3} h}$$

Example:



Find CoM.

Solution:

$\bar{x}^{rect} = -1, \quad \bar{y}^{rect} = 2, \quad m^{rect} = 8\rho$

$\bar{x}^{tri} = 4/3, \quad \bar{y}^{tri} = 4/3, \quad m^{tri} = 8\rho$

$M_y^{rect} = -8\rho = \bar{x}^{rect} \cdot m^{rect}, \quad M_x^{rect} = 16\rho$

$M_y^{tri} = \frac{32}{3}\rho = \bar{y}^{tri} \cdot m^{tri}, \quad M_x^{tri} = \frac{32}{3}\rho$

$M_y^{tot} = \frac{8}{3}\rho, \quad M_x^{tot} = \frac{80}{3}\rho, \quad m^{tot} = 16\rho$

$\rightsquigarrow \quad \bar{x}^{tot} = \frac{8\rho/3}{16\rho} = \frac{1}{6}, \quad \bar{y}^{tot} = \frac{80\rho/3}{16\rho} = \frac{5}{3}$

$$\text{CoM} = \boxed{\left(\frac{1}{6}, \frac{5}{3}\right)}$$

2/18

Improper Integrals

(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ (b) $\displaystyle\int_0^2 \frac{1}{x}\,dx$ (c) $\displaystyle\int_{-1}^{+1} \frac{1}{x^2}\,dx$

$\frac{1}{0^+} = \infty$ $\frac{1}{0} = \text{DNE}$

Method / Definition: convert to limit

(a) $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R x^{-2}\,dx$$

$$\rightsquigarrow \lim_{R\to\infty}\left(\frac{x^{-1}}{-1}\Bigg|_1^R\right) \rightsquigarrow \lim_{R\to\infty}\left(\frac{-1}{R} - \frac{-1}{1}\right) = \boxed{+1,\ \text{converges}}$$

(b) $$\int_0^2 \frac{1}{x}\,dx = \lim_{R\to 0^+} \int_R^2 x^{-1}\,dx$$

Example "2^-": $\displaystyle\int_0^2 \frac{1}{x-2}\,dx = \lim_{R\to 2^-} \int_0^R \frac{1}{x-2}\,dx$

$$\rightsquigarrow \lim_{R\to 0^+}\left(\ln x\Big|_R^2\right) \rightsquigarrow \lim_{R\to 0^+}\left(\ln 2 - \ln R\right)$$

$$\rightsquigarrow \ln 2 - -\infty$$
$$= \boxed{+\infty,\ \text{diverges}}$$

preliminary step: break @ $x=0$

(c) $$\int_{-1}^{+1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$$

$$= \lim_{R\to 0^-} \int_{-1}^R x^{-2}\,dx + \lim_{R\to 0^+} \int_R^1 x^{-2}\,dx$$

$$\rightsquigarrow \lim_{R\to 0^-}\left(\frac{R^{-1}}{-1} - \frac{(-1)^{-1}}{-1}\right) + \lim_{R\to 0^+}\left(\frac{(1)^{-1}}{-1} - \frac{R^{-1}}{-1}\right) \rightsquigarrow (+\infty - 1) + (-1 + \infty) = \boxed{\begin{array}{c}\infty\\ \text{diverges}\end{array}}$$

Example: $\int_0^9 \frac{dx}{\sqrt{x}} = ?$

Solution:

$$= \lim_{R\to 0^+} \int_R^9 x^{-1/2}\,dx \rightsquigarrow \lim_{R\to 0^+} \left.\frac{x^{1/2}}{1/2}\right|_R^9$$

$$\rightsquigarrow \lim_{R\to 0^+} \left(2\cdot 9^{1/2} - 2\cdot R^{1/2}\right) \rightsquigarrow \boxed{6, \text{ converges}}$$

$$\lim_{R\to 0^+} 2\cdot R^{1/2}$$

$$2\sqrt{R} \xrightarrow[R\to 0]{} 0$$

Example: $\int_{-1}^{+1} \frac{1}{x}\,dx = \lim_{R\to 0^-} \int_{-1}^{R} x^{-1}\,dx + \lim_{R\to 0^+} \int_R^1 x^{-1}\,dx$

$$\rightsquigarrow \lim_{R\to 0^-}\left(\ln|R| - \ln|-1|\right) + \lim_{R\to 0^+}\left(\ln|1| - \ln|R|\right)$$

$$\rightsquigarrow (-\infty - 0) + (0 - -\infty) = -\infty + \infty = \boxed{\text{DNE diverges}}$$

p-integrals:

$0 < p < \infty \qquad 0 < a < \infty$

$p > 1$: $\displaystyle\int_a^\infty \frac{dx}{x^p}$ converges $\qquad \displaystyle\int_0^a \frac{dx}{x^p}$ diverges

$p < 1$: $\displaystyle\int_a^\infty \frac{dx}{x^p}$ diverges $\qquad \displaystyle\int_0^a \frac{dx}{x^p}$ converges

$p = 1$: $\displaystyle\int_a^\infty \frac{dx}{x}$ diverges $\qquad \displaystyle\int_0^a \frac{dx}{x}$ diverges

Example: $\displaystyle\int_5^\infty \frac{1}{x^{1.2}}\,dx$ conv. $\qquad \displaystyle\int_0^3 \frac{1}{x^{0.9}}\,dx$ conv.

$\displaystyle\int_{10}^\infty \frac{1}{\sqrt{x}}\,dx$ div. $\qquad$ etc.

Comparison:

$f(x) \le g(x)$

$\displaystyle\int_a^b g\,dx$ conv. $\Rightarrow$ $\displaystyle\int_a^b f\,dx$ conv.

$\displaystyle\int_a^b f\,dx$ div. $\Rightarrow$ $\displaystyle\int_a^b g\,dx$ div.