W09 - Notes

Simple divergence test

Videos

Videos, Math Dr. Bob

- Geometric series and SDT, again: Geometric series, Simple Divergence Test (aka "Limit Test")
- Integral test: Basics
- Integral test: p-series
  - Extra: Integral test: Further examples
  - Extra: Integral test: Estimations

01 Theory

Simple Divergence Test (SDT)

Applicability: *Any* series.

Test Statement:

$$\lim_{n\to\infty} a_n \neq 0 \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

AKA the **"Not Even Close"** test

The *converse is not valid*. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

02 Illustration

Example - Simple Divergence Test: examples

(a) Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

(b) Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.

We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \ldots$," but that is not a limit value.

Positive series

Videos

- Direct Comparison Test: Theory and basic examples
- Direct Comparison Test: Series $\frac{1}{\ln n}$
- Limit Comparison Test: Theory and basic examples
- Limit Comparison Test: Further examples

03 Theory

Positive series

A series is called **positive** when its individual terms are positive, i.e. $a_n > 0$ for all n.

The partial sum sequence S_N is *monotone increasing* for a positive series.

By the monotonicity test for convergence of sequences, S_N therefore converges whenever it is *bounded above*. If S_N is not bounded above, then $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

Another test, called the **integral test**, studies the terms of a series as if they represent rectangles with upper corner pinned to the graph of a continuous function.

To apply the test, we must convert the integer index variable n in a_n into a continuous variable x . This is easy when we have a formula for a_n (provided it doesn't contain factorials or other elements dependent on integrality).

Integral Test (IT)

Applicability: $f(x)$ must be:

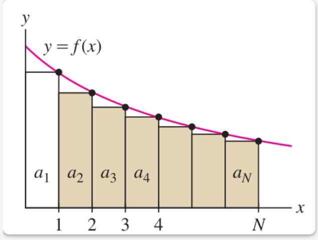
- Continuous
- Positive
- *Monotone decreasing*

Test Statement:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \qquad \Longleftrightarrow \qquad \int_1^{\infty} f(x)\,dx \quad \text{converges}$$

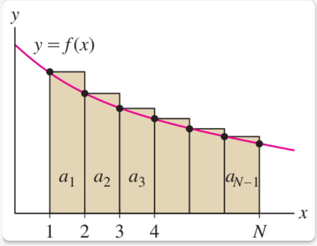
Extra - Integral test: explanation

To show that *integral convergence implies series convergence*, consider the diagram:



This shows that $\sum_{n=2}^{N} a_n \leq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ converges, then $\int_1^N f(x)\,dx$ is bounded (independent of N) and so $\sum_{n=2}^{N} a_n$ is bounded by that inequality. But $\sum_{n=2}^{N} a_n = S_N - a_1$; so by adding a_1 to the bound, we see that S_N itself is bounded, which implies that $\sum_{n=1}^{\infty} a_n$ converges.

To show that *integral divergence implies series divergence*, consider a similar diagram:



This shows that $\sum_{n=1}^{N-1} a_n \geq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^N f(x)\,dx$ goes to $+\infty$ as $N \to \infty$, and so $\sum_{n=1}^{N-1} a_n$ goes to $+\infty$ as well. So $\sum_{n=1}^{\infty} a_n$ diverges.

Notice: the picture shows $f(x)$ entirely above (or below) the rectangles. This depends upon $f(x)$ being monotone decreasing, as well as $f(x) > 0$. (This explains the applicability conditions.)

Next we use the integral test to evaluate the family of ***p*-series**, and later we can use p-series in comparison tests without repeating the work of the integral test.

p-series

A ***p*-series** is a series of this form: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

Convergence properties:

$$p > 1 : \text{series converges} \qquad\qquad p \leq 1 : \text{series diverges}$$

Extra - Proof of p-series convergence

(1) To verify the convergence properties of p-series, apply the integral test:

Applicability: verify it's continuous, positive, decreasing.

Convert n to x to obtain the function $f(x) = \frac{1}{x^p}$.

Indeed $\frac{1}{x^p}$ is continuous and positive and decreasing as x increases.

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(2) Apply the integral test.

Integrate, assuming $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \frac{x^{p-1}}{p-1}\bigg|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

When $p > 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = 0$

When $p < 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = \infty$

When $p = 1$, integrate a second time:

$$\int_1^\infty \frac{1}{x}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \ln x\bigg|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \ln R - \ln 1 \quad \gg\gg \quad \infty$$

Conclude: the integral converges when $p > 1$ and diverges when $p \leq 1$.

We could instead immediately refer to the convergence results for *p-integrals* instead of reproving them here.

04 Illustration

Example - p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^\infty \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p \leq 1$

Example - Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ converge?

Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n \ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

(1) The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.

Check applicability.

Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty)$.

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(2) Apply the integral test to $f(x)$.

Integrate $f(x)$:

$$\int_2^\infty \frac{1}{x \ln x}\, dx \quad \gg\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u}\, du$$

$$\gg\gg \quad \lim_{R\to\infty} \ln u \Big|_{\ln 1}^{R} \quad \gg\gg \quad \infty$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n \ln n}$ *diverges*.

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(3) Apply the integral test to $g(x)$.

Integrate $g(x)$:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx \quad \gg\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u^2}\, du$$

$$\gg\gg \quad \lim_{R\to\infty} -u^{-1} \Big|_{\ln 2}^{R} \quad \gg\gg \quad \frac{1}{\ln 2}$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ *converges*.

05 Theory

Direct Comparison Test (DCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose $a_n \leq b_n$ for large enough n.
(Meaning: for $n \geq N$ with some given N.) Then:

- Smaller pushes up bigger:

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} b_n \quad \text{diverges}$$

- Bigger controls smaller:

$$\sum_{n=1}^{\infty} b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{converges}$$

06 Illustration

Example - Direct comparison test: rational functions

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 3^n}$

Choose: $a_n = \frac{1}{\sqrt{n}\, 3^n}$ and $b_n = \frac{1}{3^n}$

Check: $\frac{1}{\sqrt{n}\, 3^n} \leq \frac{1}{3^n}$

Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

Therefore: *converges* by the DCT.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$

Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.

Check: $\frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$

Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

Therefore: *converges* by the DCT.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$

Check: $\frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)

Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

Therefore: *converges* by the DCT.

(d) $\sum_{n=2}^{\infty} \frac{1}{n-1}$

Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$

Check: $\frac{1}{n} \leq \frac{1}{n-1}$

Observe: $\sum \frac{1}{n}$ is a divergent p-series

Therefore: *diverges* by the DCT.

07 Theory

Some series can be compared using the DCT after applying certain manipulations and tricks.

For example, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. We suspect convergence because $a_n \approx \frac{1}{n^2}$ for *large* n. But unfortunately, $a_n > \frac{1}{n^2}$ always, so we cannot apply the DCT.

We could make some *ad hoc* arguments that do use the DCT, eventually:

Trick Method 1:

- Observe that for $n > 1$ we have $\frac{1}{n^2-1} \leq \frac{10}{n^2}$. (Check it!)
- But $\sum \frac{10}{n^2}$ converges, indeed its value is $10 \cdot \sum \frac{1}{n^2}$, which is $\frac{10\pi^2}{6}$.
- So the series $\sum \frac{1}{n^2-1}$ converges.

Trick Method 2:

- Observe that we can change the letter n to $n+1$ by starting the new n at $n = 1$.
- Then we have:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

- This last series has terms smaller than $\frac{1}{n^2}$ so the DCT with $b_n = \frac{1}{n^2}$ (a convergent p-series) shows that the original series converges too.

These convoluted arguments suggest that a more general version of Comparison is possible.

Indeed, it is sufficient to compare the *relative large-n behavior* of the two series. We use the *termwise ratios* to estimate comparative behavior for increasing n.

Limit Comparison Test (LCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose that $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$. Then:
If $0 < L < \infty$, i.e. *finite non-zero*, then:

$$\sum a_n \quad \text{converges} \qquad \iff \qquad \sum b_n \quad \text{converges}$$

Extra - LCT edge cases

If $L = 0$ or $L = \infty$, we can still draw an inference, but only in one direction:

- If $L = 0$:

$$\sum b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{converges}$$

- If $L = \infty$:

$$\sum b_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{diverges}$$

Extra - Limit Comparison Test explanation

Suppose $a_n/b_n \to L$ and $0 < L < \infty$. Then for n sufficiently large, we know $a_n/b_n < L + 1$.

Doing some algebra, we get $a_n < (L+1)b_n$ for n large.

If $\sum b_n$ converges, then $\sum (L+1)b_n$ also converges (constant multiple), and then the DCT implies that $\sum a_n$ converges.

Conversely: we also know that $b_n/a_n \to 1/L$, so $b_n < (1/L + 1)a_n$ for all n sufficiently large. Thus if $\sum a_n$ converges, $\sum (1/L+1)a_n$ also converges, and by the DCT again $\sum b_n$ converges too.

The cases with $L = 0$ or $L = \infty$ are handled similarly.

08 Illustration

Example - Limit Comparison Test examples

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Choose: $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty} \frac{2^n}{2^n - 1} \quad \ggg \quad 1 \;=:\; L$$

Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

Therefore: *converges* by the LCT.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$

Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = \frac{1}{n^{1/2}}$

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \overset{n\to\infty}{\longrightarrow} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2 \;=:\; L$$

Observe: $\sum \frac{1}{n^{1/2}}$ is a divergent p-series

Therefore: *diverges* by the LCT.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$

Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = \frac{1}{n^2}$

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{n^4}{n^4-n-1} \quad \gg\gg \quad 1 \;=:\; L$$

Observe: $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a converging p-series

Therefore: *converges* by the LCT.

Alternating series

Videos

Videos, Math Dr. Bob:

- Alternating Series Test: Theory and basic examples
- Alternating Series Test: Remainder estimates
- Alternating Series Test: Further remainder estimates

09 Theory

Consider these series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad \infty$$

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\cdots \quad = \quad -\infty$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots \quad = \quad \ln 2$$

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad ?$$

The absolute values of terms are the same between these series, only the signs of terms change.

The first is a **positive series** because there are no negative terms.

The second series is the negation of a positive series – the study of such series is equivalent to that of positive series, just add a negative sign everywhere. These signs can be factored out of the series. (For example $\sum -\frac{1}{n} = -\sum \frac{1}{n}$.)

The third series is an **alternating series** because the signs alternate in a strict pattern, every other sign being negative.

The fourth series is *not* alternating, nor is it positive, nor negative: it has a mysterious or unknown pattern of signs.

A series with any negative signs present, call it $\sum_{n=1}^{\infty} a_n$, **converges absolutely** when the positive series of absolute values of terms, namely $\sum_{n=1}^{\infty} |a_n|$, converges.

THEOREM: Absolute implies ordinary

If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it also converges as it stands.

A series might converge due to the presence of negative terms and yet *not* converge absolutely:

A series $\sum_{n=1}^{\infty} a_n$ is said to be **converge conditionally** when the series converges as it stands, but the series produced by inserting absolute values, namely $\sum_{n=1}^{\infty} |a_n|$, diverges.

The alternating harmonic series above, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \;=\; \ln 2$, is therefore conditionally convergent. Let us see why it converges. We can group the terms to create new sequences of *pairs*, each pair being a positive term. This can be done in two ways. The first creates an increasing sequence, the second a decreasing sequence:

$$\text{increasing from 0:} \qquad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots$$

$$\text{decreasing from 1:} \qquad 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots$$

Suppose S_N gives the sequence of partial sums of the original series. Then S_{2N} gives the first sequence of pairs, namely $S_2, S_4, S_6, \ldots$. And S_{2N-1} gives the second sequence of pairs, namely $S_1, S_3, S_5, \ldots$.

The second sequence shows that S_N is bounded above by 1, so S_{2N} is monotone increasing and bounded above, so it converges. Similarly S_{2N-1} is monotone decreasing and bounded below, so it converges too, and of course they must converge to the same thing.

The fact that the terms were *decreasing in magnitude* was an essential ingredient of the argument for convergence. This fact ensured that the parenthetical pairs were positive numbers.

Alternating Series Test (AST) - "Leibniz Test"

Applicability: Alternating series only: $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $a_n > 0$

Test Statement:
If:

1. $a_n \to 0$ as $n \to \infty$ (i.e. it passes the SDT: if this fails, conclude *diverges*)
2. a_n are *decreasing*, so $a_1 > a_2 > a_3 > a_4 > \cdots > 0$

Then:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{converges}$$

"Next Term Bound" rule for error of the partial sums:

$$|S - S_N| \leq a_{N+1}$$

Extra - Alternating Series Test: Theory

Just as for the alternating harmonic series, we can form *positive* paired-up series because the terms are decreasing:

$$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots$$

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots$$

The first sequence S_{2N} is monotone increasing from 0, and the second S_{2N-1} is decreasing from a_1. The first is therefore also bounded above by a_1. So it converges. Similarly, the second converges. Their difference at any point is $S_{2N} - S_{2N-1}$ which is equal to $-a_{2N}$, and this goes to zero. So the two sequences must converge to the same thing.

By considering these paired-up sequences and the effect of adding each new term one after the other, we obtain the following order relations:

$$0 < S_2 < S_4 < S_6 < \ \cdots \ < S < \ \cdots \ < S_5 < S_3 < S_1 = a_1$$

Thus, for *any even* $2N$ and *any odd* $2M-1$:

$$S_{2N} < S < S_{2M-1}$$

Now set $M = N$ and subtract S_{2N-1} from both sides:

$$S_{2N} - S_{2N-1} < S - S_{2N-1} < 0$$

$$\gg\gg \quad -a_{2N} < S - S_{2N-1} < 0$$

Now set $M = N + 1$ and subtract S_{2N} from both sides:

$$0 < S - S_{2N} < S_{2N+1} - S_{2N}$$

$$\gg\gg \quad 0 < S - S_{2N} < a_{2N+1}$$

This covers both even cases ($n = 2N$) and odd cases ($n = 2N - 1$). In either case, we have:

$$|S - S_n| < a_{n+1}$$

10 Illustration

Example - Alternating Series Test: Basic illustration

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.

Therefore the first series converges *conditionally*.

(b) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converges by the AST.

Notice the funny notation: $\cos n\pi = (-1)^n$.

This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Example - Approximating π

The Taylor series for $\tan^{-1} x$ is given by:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate π with an error less than 0.001.

Solution

(1) The main idea is to use $\tan \frac{\pi}{4} = 1$ and thus $\tan^{-1} 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

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(2) Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. Therefore, calculate n such that $a_{n+1} < 0.001$, and then we will know:

$$|E_n| < a_{n+1} < 0.001$$

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(3) The general term is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Now solve:

$$a_{n+1} = \frac{4}{2n+1} < 0.001$$

$$\gg\gg \qquad \frac{4}{0.001} < 2n+1$$

$$\gg\gg \qquad 3999 < 2n$$

$$\gg\gg \qquad 2000 \leq n$$

We conclude that at least 2000 terms are necessary to be confident (by the error formula) that the approximation of π is accurate to within 0.001.