Series Convergence

$\sum_{n=0}^{\infty} a_n$ converges means: S_N converges

$$S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \ldots + a_N$$

S_N = "partial sum sequence of the series"

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$S_3 = a_0 + a_1 + a_2 + a_3$$
$$S_4 = a_0 + \ldots + a_4$$
$$\vdots$$
$$\lim_{N\to\infty} S_N \overset{\text{(def.)}}{=} \sum_{n=0}^{\infty} a_n$$

Geometric Series

$$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r} = \text{"}S\text{"}$$

$$S_N = a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \ldots + a_0 r^N$$
$$-\left(r S_N = a_0 r + a_0 r^2 + a_0 r^3 + \ldots + a_0 r^N + a_0 r^{N+1}\right)$$

$$S_N - r S_N = a_0 - a_0 r^{N+1}$$

$$\rightsquigarrow S_N = \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1}$$

($N+1 \to \infty$)

Therefore: $|r| < 1 \Rightarrow S_N \to \frac{a_0}{1-r}$ as $N \to \infty$

$|r| \geq 1 \Rightarrow S_N \to \pm\infty$, DNE as $N \to \infty$

Geometric Series Test:

Write series as $\sum_{n=0}^{\infty} a_0 r^n$ i.e. as geometric in standard form

Then: $|r| < 1 \rightsquigarrow$ converges
$|r| \geq 1 \rightsquigarrow$ diverges

Moreover: $\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$ when $|r| < 1$.

Simple Divergence Test

Applicability: any series

Test: $\lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

AKA: "Not even close" test...

Note: Converse not true: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges "harmonic series"

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \to \infty$

$\sum \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \to \ln 2$? (finite)

However, converse holds for geometric series:

$\sum a_0 r^n$: $|r| < 1 \rightsquigarrow$ converges AND $a_0 r^n \to 0$
$|r| \geq 1 \rightsquigarrow$ diverges AND $a_0 r^n \to \neq 0$

Examples: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

converge or diverge??

Solution: (a) $a_n = \frac{n}{4n+1} \to \frac{1}{4} \neq 0$

Therefore series diverges by SDT.

(b) $\frac{n}{n+1} \to 1$ so a_n " $\longrightarrow$ " $+1, -1, +1, -1, \ldots$

i.e. $\lim a_n = \text{DNE} \neq 0$

Therefore series diverges by SDT.

Positive Series

i.e. $a_n > 0$ all n

Theory: S_N is monotone increasing, so by Monotonicity Thm:

$\rightsquigarrow \sum a_n$ converges $\iff$ S_N is bounded.

Integral Test

Setup: write $f(x) = a_x$

Applicability:

1. f is continuous
2. $f \geq 0$
3. f is monotone decreasing

Test: $\displaystyle\sum_{n=1}^{\infty} a_n$ conv./div. $\iff$ $\displaystyle\int_1^{\infty} f(x)\,dx$ conv./div.

don't care about starting value/lower bound

same convergence
not equal values

Examples: (a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution:

(a) $f(x) = \frac{1}{x \ln x}$

1. continuous ✓
2. $f(x) \geq 0$ when $x > 1$ ✓
3. f decreasing ??

Argument 1: "Both denom factors growing"

Argument 2:

$$f'(x) = \frac{(x \ln x)(0) - (1)(1 \cdot \ln x + x \cdot 1/x)}{(x \ln x)^2}$$

$$= -\frac{\ln x + 1}{(x \ln x)^2} < 0 \quad \text{when } x > 1$$

Hence decreasing when $x > 1$ ✓ (know $x \geq 2$)

Test:

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx \xrightarrow[du = \frac{1}{x}dx]{u = \ln x} \int_{\ln 2}^{\infty} \frac{1}{u}\, du$$

$$\rightsquigarrow \lim_{R \to \infty} \int_{\ln 2}^{R} \frac{1}{u}\, du \rightsquigarrow \lim_{R \to \infty} \left(\ln R - \ln \ln 2 \right)$$

$$= \infty$$

Therefore **diverges.**

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge or diverge?

Integral test applicability? $f(x) = \frac{1}{x(\ln x)^2}$

1. continuous ✓
2. positive ✓ $(x>1)$
3. decreasing?

Argument 1: all 3 denom. factors are increasing

Argument 2: differentiate:

$$f'(x) = \frac{(x(\ln x)^2)\cdot 0 - 1\cdot(1\cdot(\ln x)^2 + x\cdot 2\cdot \ln x\cdot 1/x)}{x^2(\ln x)^4}$$

$$= -\frac{(\ln x)^2 + 2\ln x}{x^2(\ln x)^2} < 0 \quad \text{when } x>1.$$

Neg. derivative $\Rightarrow$ decreasing ✓

Apply IT:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ conv/div} \iff \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \text{ conv/div.}$$

Do integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^{R} \frac{1}{x(\ln x)^2}\,dx \underset{du=\frac{1}{x}dx}{\overset{u=\ln x}{\leadsto}} \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{u^2}\,du$$

$$\leadsto \lim_{R\to\infty} \left(\frac{(\ln R)^{-1}}{-1} - \frac{(\ln 2)^{-1}}{-1} \right) = \frac{1}{\ln 2} \quad \text{thus}$$

$$\frac{1}{\ln 2} - \frac{1}{\ln R}$$

integral AND **series converge.**

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ — diverges for $p \le 1$; converges for $1 < p$

Example: (a) $\sum \frac{1}{n^2}$ converges

(b) $\sum \frac{1}{\sqrt{n}}$ diverges

Why? IT:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big|_1^R\right)$$

$$= \lim_{R\to\infty}\left(\frac{R^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

$$= \frac{1}{p-1} \quad \text{if } p > 1$$

$$= \infty \quad \text{else}$$

Direct Comparison Test

Applicability: both series positive: $a_n > 0$, $b_n > 0$ all n

Test: Suppose $a_n \le b_n$ for all n. Then:

- $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges

  "smaller pushes up bigger"

- $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

  "bigger controls smaller"

Examples: (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 3^n}$

Notice: $\left(\frac{1}{3}\right)^n$ is converging geometric.

Observe: $\frac{1}{\sqrt{n}}\left(\frac{1}{3}\right)^n \le \left(\frac{1}{3}\right)^n$

$$\underset{a_n}{\shortparallel} \qquad\qquad \underset{b_n}{\shortparallel}$$

Therefore: $\sum b_n$ $\left(= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n\right)$ converges $\overset{\text{DCT}}{\Rightarrow}$ $\boxed{\sum a_n \text{ converges}}$

(b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ $\rightsquigarrow$ $\sum_{n=1}^{\infty} (\cos^2 n)\left(\frac{1}{n^3}\right)$

Notice: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $\left(\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty}\left(\frac{R^{-2}}{-2} - \frac{1^{-2}}{-2}\right) = \frac{1}{2} \right)$

Observe: $(\cos^2 n)\left(\frac{1}{n^3}\right) \le \frac{1}{n^3}$ bc $0 \le \cos^2 n \le 1$

$$\underset{a_n}{\shortparallel} \qquad\qquad \underset{b_n}{\shortparallel}$$

DCT $\rightsquigarrow$ $\boxed{\text{converges}}$

DCT Examples Ct'd

(c) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ Check positive ✓

Notice: $\underbrace{\frac{n}{n^3+1}}_{a_n} \leq \frac{n}{n^3} = \underbrace{\frac{1}{n^2}}_{b_n}$

Have $\sum \frac{1}{n^2}$ is p-series with $p = 2 > 1$ ⤳ conv.

So by DCT: $\boxed{\sum_{n=1}^{\infty} \frac{n}{n^3+1} \text{ converges}}$

(d) $\sum_{n=2}^{\infty} \frac{1}{n-1}$ Check positive ✓

Notice: $\underbrace{\frac{1}{n-1}}_{a_n} \geq \underbrace{\frac{1}{n}}_{b_n}$

Have $\sum \frac{1}{n}$ diverges, p-series, $p = 1$.

By DCT: $\boxed{\sum \frac{1}{n-1} \text{ also diverges}}$

Limit Comparison Test

E.g. ① $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ or ② $\sum_{n=2}^{\infty} \frac{1}{n+1}$

① Try: Notice $\frac{n}{n^3-1} \geq \frac{n}{n^3} = \frac{1}{n^2}$

Have $\sum \frac{1}{n^2}$ conv. $(p=2)$ $\overset{DCT}{\Rightarrow}$ nothing!

② Try: Notice $\frac{1}{n+1} \leq \frac{1}{n}$

Have $\sum \frac{1}{n}$ div. $(p=1)$ $\overset{DCT}{\Rightarrow}$ nothing!

Applicability: $a_n > 0$, $b_n > 0$ all n

Test: $\frac{a_n}{b_n} \overset{?}{\underset{?}{\longrightarrow}} L$ as $n \to \infty$

$$\frac{a_0 + a_1 + a_2 + a_3 + \dots}{b_0 + b_1 + b_2 + b_3 + \dots} \rightsquigarrow \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots \overset{?}{\underset{?}{\longrightarrow}} L$$

If $0 < L < \infty$ (i.e. finite nonzero), then:

$\sum a_n$ conv./div. $\Leftrightarrow$ $\sum b_n$ conv./div.

Examples:

from above:

$\frac{3n^2}{3n^2} = 1$ ↑ L'H

① $\sum_{n=2}^{\infty} \frac{n}{n^3-1} \rightsquigarrow \frac{a_n = \frac{n}{n^3-1}}{b_n = \frac{n}{n^3}} = \frac{n}{n^3-1} \cdot \frac{n^3}{n} \rightsquigarrow \frac{n^3}{n^3-1} \to 1$

So $L=1$, $0 < L < \infty$ so by LCT both conv./div.

Have $\sum_{n=2}^{\infty} \frac{n}{n^3}$ conv. $(p=2)$ i.e. $\sum \frac{1}{n^2}$

So by LCT $\boxed{\sum_{n=2}^{\infty} \frac{n}{n^3-1} \text{ converges}}$

② $\sum_{n=2}^{\infty} \frac{1}{n+1} \rightsquigarrow \frac{a_n = \frac{1}{n+1}}{b_n = \frac{1}{n}} = \frac{1}{n+1} \cdot \frac{n}{1} \rightsquigarrow \frac{n}{n+1} \xrightarrow{n\to\infty} 1$

So $L=1$ and $0 < L < \infty$, so LCT ✓

Have $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges $(p=1)$.

So by LCT, $\boxed{\sum_{n=2}^{\infty} \frac{1}{n+1} \text{ also diverges}}$

Example:

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \rightsquigarrow \frac{a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}}{b_n = \frac{2n^2}{\sqrt{n^5}}} = \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{\sqrt{n^5}}{2n^2}$$

$$\frac{\sqrt{n^5}}{2n^2} = \frac{\sqrt{n}}{2}$$

$$\rightsquigarrow \frac{2n^{\frac{5}{2}} + 3n^{\frac{3}{2}}}{2\sqrt{5+n^5}} \longrightarrow \frac{2n^{\frac{5}{2}}}{2\sqrt{n^5}} \longrightarrow 1 \quad \text{as } n \to \infty$$

So $L = 1$, $0 < L < \infty$, $\sum \frac{2}{\sqrt{n}}$ div. ($p = 1/2$)

Therefore by LCT, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges.

Alternating Series

e.g. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \pi/4$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots = \ln 2$

Write $\sum_{n=0}^{\infty} (-1)^n a_n \qquad a_n > 0$

Applicability: any alternating series

Test: 1. Pass SDT: $a_n \to 0$ as $n \to \infty$

2. $a_0 > a_1 > a_2 > a_3 > a_4 > \dots$ i.e. decreasing

Then $\sum (-1)^n a_n$ converges.

Furthermore: $|S - S_N| < a_{N+1}$ ("error")

"Next Term Error Bound" formula

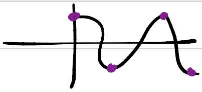
Examples: (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$

Solution: (a) • $a_n = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ ✓

• decreasing ✓ ($\sqrt{n}$ increasing)

$\Rightarrow$ By AST it **converges.**

(b) $\cos(n\pi) = (-1)^n$ so it's alternating.



• $a_n = \frac{1}{n^2} \to 0$ ✓ • decreasing ✓ $\Rightarrow$ By AST it **converges**

Example: Fact: $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$

How many terms needed to approximate π to within 0.001?

Solution: Notice that $\tan^{-1}(1) = \frac{\pi}{4}$, so:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \dots + \frac{4}{2n+1}$$

Apply "Next Term Bound": find N to get:

$a_{N+1} < 0.001$ because then:

$$|s - s_N| < a_{N+1} < 0.001$$

desired inequality

Set $\frac{4}{2n+1} < 0.001$

$\leadsto$ $4{,}000 < 2n+1$

$\leadsto$ $2{,}000 - \frac{1}{2} < n$

$\leadsto$ $n = 2{,}000$ or larger