# Ratio Test

Applicability: Any series $\sum a_n$ with $a_n \neq 0$ all $n$.

Test: limit of sequence of successive term ratios:

$$\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow{n \to \infty} L$$

Then: $L < 1 \rightsquigarrow \sum_{n=0}^{\infty} a_n$ converges absolutely

$L > 1 \rightsquigarrow \sum_{n=0}^{\infty} a_n$ diverges

$L = 1$, DNE $\rightsquigarrow$ test inconclusive

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**Example:** $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!} \rightsquigarrow \left| \frac{a_{n+1} = \frac{10^{n+1}}{(n+1)!}}{a_n = \frac{10^n}{n!}} \right|$

$$\rightsquigarrow \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \rightsquigarrow \frac{10 \cdot \cancel{10^n}}{(n+1)\,\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{10^n}} \rightsquigarrow \frac{10}{n+1} \xrightarrow{n \to \infty} 0$$

$$(n+1)! = (n+1)\underbrace{(n)(n-1)(n-2)\cdots(3)(2)(1)}_{n!} = (n+1)\,n!$$

$3! = 3 \cdot 2 \cdot 1$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$n! = (n)(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

So $L = 0$, $L < 1$

$$\rightsquigarrow \boxed{\sum_{n=0}^{\infty} \frac{10^n}{n!} \text{ converges by RaT.}}$$

**Example:** $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} \rightsquigarrow \left| \frac{a_{n+1} = \frac{(n+1)^2}{2^{n+1}}}{a_n = \frac{n^2}{2^n}} \right|$

$$\rightsquigarrow \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \rightsquigarrow \frac{n^2+2n+1}{2 \cdot \cancel{2^n}} \cdot \frac{\cancel{2^n}}{n^2} \qquad 2^{n+1} = 2^n \cdot 2^1$$

$$\rightsquigarrow \frac{n^2+2n+1}{2n^2} \xrightarrow{n \to \infty} \frac{1}{2} = L$$

So $L < 1 \rightsquigarrow$ $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} \text{ converges absolutely by Ratio Test}}$

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**Notice:** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge $(p = 2 > 1)$

Yet: $\left| \frac{a_{n+1} = \frac{1}{(n+1)^2}}{a_n = \frac{1}{n^2}} \right| \rightsquigarrow \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} \rightsquigarrow \frac{n^2}{n^2+2n+1} \xrightarrow{n \to \infty} 1$

So $L = 1 \rightsquigarrow$ RaT is inconclusive.

**Also:** $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges (fails SDT)

Yet: $\left| \frac{a_{n+1}}{a_n} \right| \rightsquigarrow \frac{(n+1)^2}{n^2} = \frac{n^2+2n+1}{n^2} \xrightarrow{n \to \infty} 1$ RaT inconclusive.

**Moral:** RaT bad for rat. functions...

# Explanation of Ratio Test:

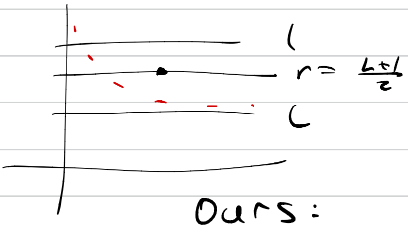
Have series $\sum_{n=0}^{\infty} a_n$, $\quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

Write $R_n = \frac{a_{n+1}}{a_n}$. So $|R_n| \to L$, and

$a_{n+1} = R_n \cdot a_n$

(e.g. set $r = \frac{L+1}{2}$)

Say $L < 1$. Pick any $r$ between: $L < r < 1$




Eventually $|R_n| < r$.

i.e. $|R_n| < r$ all $n > N$ (some $N$).

Ours:

$$a_0 = a_0$$
$$R_0 a_0 = a_1$$
$$R_1 R_0 a_0 = a_2$$
$$R_2 R_1 R_0 a_0 = a_3$$
$$\vdots$$

Geometric:

$$a_0 = a_0$$
$$r a_0 = a_0 r$$
$$r \cdot r \cdot a_0 = a_0 r^2$$
$$r \cdot r \cdot r \cdot a_0 = a_0 r^3$$
$$\vdots$$

$r < 1$, $\sum_{n=N}^{\infty} a_0 r^n$ converges

But $|a_n| < a_0 r^n$ $\quad n > N$ (Direct Comparison)

so $\sum_{n=N}^{\infty} |a_n|$ converges

$\rightsquigarrow \sum_{n=0}^{\infty} a_n$ converges absolutely.

Illustration: $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!}$

$$R_n = \frac{10}{n+1} \qquad \text{when} \quad n > \overset{\text{"}N\text{"}}{\overset{\shortparallel}{20}}, \quad R_n < \frac{1}{2}$$

so set $r = \frac{1}{2}$

$$\sum_{n=21}^{\infty} \frac{10^n}{n!} \leq \sum_{n=21}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^n$$

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## Root Test

Applicability: Any series.

Test: $$\underbrace{|a_n|^{1/n}}_{= \sqrt[n]{|a_n|}} \xrightarrow{n \to \infty} L$$

Then:

$L < 1 \rightsquigarrow \displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely

$L > 1 \rightsquigarrow \displaystyle\sum_{n=0}^{\infty} a_n$ diverges

$L = 1$, DNE $\rightsquigarrow$ test inconclusive

## Explanation of Root:

Suppose $|a_n|^{1/n} \to L < r < 1$

Eventually $(n > N)$, $|a_n|^{1/n} < r$ all $n > N$

$\Downarrow$

$|a_n| < r^n$ all $n > N$

$\sum_{n=N+1}^{\infty} r^n$ converges (Geometric with $r < 1$)

$\rightsquigarrow \sum_{n=N+1}^{\infty} |a_n|$ converges by Direct Comparison

$\rightsquigarrow \sum_{n=0}^{\infty} a_n$ converges absolutely.

Examples: (a) $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ (b) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^n$

Solutions:

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n \rightsquigarrow \left|a_n=\left(\frac{1}{n}\right)^n\right|^{1/n} \rightsquigarrow \left(\left(\frac{1}{n}\right)^n\right)^{1/n}$

$\rightsquigarrow \left(\frac{1}{n}\right)^{n\cdot\frac{1}{n}} \rightsquigarrow \frac{1}{n} \xrightarrow{n\to\infty} 0$

So $L=0$, $L<1$ $\rightsquigarrow$ $\boxed{\text{by Root, converges absolutely}}$

(b) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^n \rightsquigarrow \left|(-1)^n\left(\frac{n}{2n+1}\right)^n\right|^{1/n}$

$\rightsquigarrow \left(\frac{n}{2n+1}\right)^{n\cdot\frac{1}{n}} \rightsquigarrow \frac{n}{2n+1} \xrightarrow{n\to\infty} \frac{1}{2}$

So $L=\frac{1}{2}$, $L<1$ $\rightsquigarrow$ $\boxed{\text{by Root, converges absolutely}}$

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You try: $\sum_{n=1}^{\infty}\frac{n^2 4^n}{5^{n+2}}$

Rat or Root ??
Do it both ways!

(before Tue. 1:00pm)

Example: $\sum_{n=1}^{\infty} \frac{n^2 4^n}{5^{n+2}} \rightsquigarrow \frac{n^2}{5^2}\cdot\left(\frac{4}{5}\right)^n$

Ratio Test:

$$\rightsquigarrow \left| \frac{a_{n+1} = \frac{(n+1)^2 4^{n+1}}{5^{n+1+2}}}{a_n = \frac{n^2 4^n}{5^{n+2}}} \right| \rightsquigarrow \frac{(n^2+2n+1)\cancel{4^n}\cdot 4}{5^{\cancel{n+3}}}\cdot\frac{\cancel{5^{n+2}}}{n^2 \cancel{4^n}} \rightsquigarrow \frac{4(n^2+2n+1)}{5\cdot n^2}$$

$$\rightsquigarrow \frac{4}{5}\left(\frac{n^2+2n+1}{n^2}\right) \xrightarrow{n\to\infty} \frac{4}{5}, \quad \text{So } L = 4/5, \; L<1$$

$$\rightsquigarrow \boxed{\text{converges absolutely by RaT.}}$$

Root Test:

$$\rightsquigarrow |a_n|^{1/n} = \left|\frac{n^2 4^n}{5^{n+2}}\right|^{1/n} \rightsquigarrow \frac{n^{2/n}\cdot 4^{n/n}}{5^{2/n}\cdot 5^{n/n}} \rightsquigarrow \left(\frac{n}{5}\right)^{2/n}\left(\frac{4}{5}\right) \longrightarrow ??$$

$\hookrightarrow \sqrt[n]{n^2/25}$

Indeterminate: $\infty^0$

Take log: $\ln\left(\left(\frac{n}{5}\right)^{2/n}\cdot\frac{4}{5}\right) = \frac{2}{n}\ln\left(\frac{n}{5}\right) + \ln\left(\frac{4}{5}\right)$

$$\rightsquigarrow \frac{2\ln(n/5)}{n} + \ln\left(\frac{4}{5}\right)$$

$$\frac{2\ln(n/5)}{n} \xrightarrow{\text{L'H}} \frac{\frac{2}{\frac{n}{5}}\cdot\frac{1}{5}}{1} \rightsquigarrow \frac{2}{n} \longrightarrow 0 \text{ as } n\to\infty$$

so $\rightarrow \ln\left(\frac{4}{5}\right)$ as $n\to\infty$.

So $e^{\ln\left(\left(\frac{n}{5}\right)^{2/n}\cdot\frac{4}{n}\right)} = \left(\frac{n}{5}\right)^{\frac{2}{n}}\left(\frac{4}{n}\right) \longrightarrow e^{\ln\left(\frac{4}{5}\right)} = \frac{4}{5}$

So $L = \frac{4}{5}$, $L<1$ $\rightsquigarrow \boxed{\text{conv. abs. by RooT.}}$

SDT

1) $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ DIV

LCT

2) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ CONV

IT

3) $\sum_{n=1}^{\infty} n e^{-n^2}$ CONV

DCT/LCT

4) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$ ABS CONV

RaT

5) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ CONV

DCT + Geo

6) $\sum_{n=1}^{\infty} \frac{1}{2+3^n} \le \left(\frac{1}{3}\right)^n$ CONV

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1) Notice: no exponents; highest pwrs same $\rightsquigarrow \frac{n}{2n} = \frac{1}{2}$

$\rightsquigarrow$ terms $\to \frac{1}{2}$ so SDT $\Rightarrow$ diverges

2) Notice: "algebraic", large $n \rightsquigarrow$ p-series

$$\rightsquigarrow \frac{a_n = \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{b_n = \frac{\sqrt{n^3}}{3n^3}} \rightsquigarrow \frac{3n^3}{3n^3+4n^2+2} \cdot \frac{\sqrt{n^3+1}}{\sqrt{n^3}} \longrightarrow 1 \cdot 1 = 1$$

So $L = 1$, $0 < L < \infty$

$$\sum b_n = \sum \frac{\sqrt{n^3}}{3n^3} \rightsquigarrow \sum \frac{1}{3} \cdot \frac{1}{n^{3/2}} = \frac{1}{3} \sum \frac{1}{n^{3/2}}$$

converges $(p = 3/2)$

So by LCT, OP converges.

3) $$\int_1^{\infty} \underbrace{x e^{-x^2} dx}_{} \xrightarrow[du = -2x\,dx]{u = -x^2} \int_{-1}^{-\infty} -\frac{1}{2} e^u \, du \rightsquigarrow \lim_{R \to -\infty} \int_{-1}^{R} -\frac{1}{2} e^u \, du$$

$$-\frac{1}{2} e^{-x^2} (-2x\,dx)$$

$$\rightsquigarrow \lim_{R \to -\infty} \left(-\frac{1}{2} e^R - -\frac{1}{2} e^{-1}\right)$$

$$= \frac{1}{2} e^{-1}$$ $\boxed{\text{converges}}$

# Power Series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

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## Radius of Convergence

Apply ratio test with "$x$" included:

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| \rightsquigarrow \left| \frac{a_{n+1}}{a_n} \right| |x| \longrightarrow \left( \lim \left| \frac{a_{n+1}}{a_n} \right| \right) |x|$$

So $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$, conv. when $L < 1$

$$\iff |x| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} \iff \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < \frac{1}{|x|}$$

"$R$" = "radius of convergence"

So
$|x| < R \rightsquigarrow$ converges
$|x| > R \rightsquigarrow$ diverges
$|x| = R \rightsquigarrow$ ?? look more closely

check "endpoints" manually

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"standard form"

**Example:** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \rightsquigarrow \sum_{n=0}^{\infty} \frac{1}{2^n} x^n$

$$\left| \frac{a_{n+1} = \frac{1}{2^{n+1}} x^{n+1}}{a_n = \frac{1}{2^n} x^n} \right| \rightsquigarrow \left| \frac{2^n}{2^{n+1}} \cdot \frac{x^{n+1}}{x^n} \right| \rightsquigarrow \frac{1}{2}|x| \xrightarrow{n \to \infty} \frac{1}{2}|x| = L$$

Ratio Test : $L < 1 \rightsquigarrow$ conv. , $L > 1$ div.

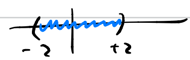
$\Longleftrightarrow$

$\frac{1}{2}|x| < 1$ conv.

$\Longleftrightarrow$

$|x| < 2$ conv., $|x| > 2$ div.

So radius of convergence is $R = 2$,




"Preliminary" interval is $I^\circ = (-2, +2)$

Actual interval : check endpoints :

@ $x = -2$ :

$$\rightsquigarrow \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n}$$

$$\rightsquigarrow \sum_{n=0}^{\infty} (-1)^n$$

Diverges (DT)

@ $x = +2$ :

$$\rightsquigarrow \sum_{n=0}^{\infty} \frac{(+2)^n}{2^n}$$

$$\rightsquigarrow \sum_{n=0}^{\infty} 1$$

Diverges (DT)

So: interval of convergence is $\boxed{I = (-2, +2)}$

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ Find $R$, $I$.

Solution: $$\left| \frac{a_{n+1} = \dfrac{x^{2(n+1)}}{(2(n+1))!}}{a_n = \dfrac{x^{2n}}{(2n)!}} \right| \rightsquigarrow \left| \frac{x^{\cancel{2n+2}}}{(2n+2)!} \cdot \frac{\cancel{(2n)!}}{\cancel{x^{2n}}} \right|$$

$$(2n+2)! = (2n+2)(2n+1)\cancel{(2n)}!$$

$$\rightsquigarrow \left| \frac{1}{(2n+2)(2n+1)} \cdot x^2 \right| \rightsquigarrow \frac{1}{(2n+2)(2n+1)} |x^2| \xrightarrow{n\to\infty} 0 = \text{"}L\text{"}$$

So by RaT: converges for all $x$.

$$\boxed{R = \infty, \quad I = (-\infty, +\infty) = \mathbb{R}}$$

Example: $\sum_{n=0}^{\infty} \frac{(-3x)^n}{\sqrt{n+1}}$ Find R, I.

Solution:

$$\left| \frac{a_{n+1} = \frac{(-3x)^{n+1}}{\sqrt{n+1+1}}}{a_n = \frac{(-3x)^n}{\sqrt{n+1}}} \right| \rightsquigarrow \left| \frac{\cancel{(-3)^n}(-3)x^{\cancel{n+1}}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n}\cancel{x^n}} \right|$$

$$\rightsquigarrow \frac{3\sqrt{n+1}}{\sqrt{n+2}}|x| \xrightarrow{n\to\infty} 3|x| = \text{"}L\text{"}$$

So by RaT: $3|x|<1 \rightsquigarrow$ conv.

Solve: $|x| < 1/3 \rightsquigarrow$ conv.

So $\boxed{R = 1/3}$, "prelim." $I^\circ = (-1/3, +1/3)$.

Check endpoints:

@ $x = -1/3$

$$\rightsquigarrow \sum_{n=0}^{\infty} \frac{(-3(-1/3))^n}{\sqrt{n+1}} \rightsquigarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \text{ div.}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div ($p = 1/2$)

$\parallel$

( OR LCT: $\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} \rightsquigarrow \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow{n\to\infty} 1$ and $\sum \frac{1}{\sqrt{n}}$ div.

So by LCT, it div. )

@ $x = +1/3$

$$\rightsquigarrow \sum_{n=0}^{\infty} \frac{(-3(1/3))^n}{\sqrt{n+1}} \rightsquigarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

conv. (cond.) by AST

$\frac{1}{\sqrt{n+1}}$ is:
- decreasing ✓
- lim to 0 ✓

So final interval: $\boxed{I = (-1/3, +1/3]}$

**Example:** $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$ Find $R$, $I$.

**Solution:**

$$\left| \frac{a_{n+1} = \dfrac{(4x+1)^{n+1}}{n+1}}{a_n = \dfrac{(4x+1)^n}{n}} \right| \leadsto \left| \frac{(4x+1)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(4x+1)^n}} \right|$$

$$\leadsto \frac{n}{n+1}|4x+1| \xrightarrow{n\to\infty} |4x+1| = \text{"}L\text{"}$$

So by RaT: $|4x+1| < 1 \leadsto$ conv.

solve: $|x-(-1/4)| < 1/4 \leadsto$ conv.

So $\boxed{R = 1/4}$, prelim. $I^\circ = \left(-\frac{1}{4}-\frac{1}{4}, -\frac{1}{4}+\frac{1}{4}\right)$

$= (-1/2, 0)$

Check endpoints:

@ $x = -1/2$:

$$\leadsto \sum_{n=1}^{\infty} \frac{(4(-1/2)+1)^n}{n} \leadsto \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ conv. by AST}$$

@ $x = 0$:

$$\leadsto \sum_{n=1}^{\infty} \frac{(4(0)+1)^n}{n} \leadsto \sum_{n=1}^{\infty} \frac{1}{n} \text{ div. by } p=1$$

So interval of convergence: $\boxed{I = [-1/2, 0)}$