W11 - Notes

Taylor and Maclaurin series

Videos

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- Maclaurin series: $f(x) = \frac{1}{(1-x)^2}$
- Maclaurin series: $f(x) = e^x$
- Maclaurin series: $f(x) = \sin x,\ \cos x,\ \tan x$
- Taylor series: $f(x) = \ln x$ at $x = 1$

01 Theory

Suppose that we have a power series function:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Consider the *successive derivatives* of f:

$$\begin{array}{rcccccccccccc}
f(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + & \cdots \\
f'(x) & = & 0 & + & a_1 & + & 2\cdot a_2 x^1 & + & 3\cdot a_3 x^2 & + & 4\cdot a_4 x^3 & + & \cdots \\
f''(x) & = & 0 & + & 0 & + & 2\cdot a_2 & + & 3\cdot 2\cdot a_3 x^1 & + & 4\cdot 3\cdot a_4 x^2 & + & \cdots \\
f'''(x) & = & 0 & + & 0 & + & 0 & + & 3\cdot 2\cdot 1\cdot a_3 & + & 4\cdot 3\cdot 2\cdot a_4 x^1 & + & \ldots \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \\
f^{(n)}(x) & = & 0 & + & 0 & + & 0 & + & 0 & + & \cdots + n!\cdot a_n & + & \cdots
\end{array}$$

When these functions are evaluated at $x = 0$, all terms with a positive x-power become zero:

$$\begin{array}{rcrcr}
f(0) & = & a_0 & = & a_0 \\
f'(0) & = & a_1 & = & a_1 \\
f''(0) & = & 2\cdot a_2 & = & 2!\cdot a_2 \\
f'''(0) & = & 3\cdot 2\cdot a_3 & = & 3!\cdot a_3 \\
\vdots & = & \vdots & = & \vdots \\
f^{(n)}(0) & = & n\cdot(n-1)\cdots 2\cdot 1\cdot a_n & = & n!\cdot a_n
\end{array}$$

This last formula is the basis for Taylor and Maclaurin series:

Power series: Derivative-Coefficient Identity

$$f^{(n)}(0) \quad = \quad n!\cdot a_n$$

This identity holds for a power series function $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ which has a nonzero radius of convergence.

We can apply the identity in both directions:

- Know $f(x)$? $\quad\rightsquigarrow\quad$ Calculate a_n for any n.

- Know a_n? $\rightsquigarrow$ Calculate $f^{(n)}(0)$ for any n.

Many functions can be 'expressed' or 'represented' near $x = c$ (i.e. for small enough $|x - c|$) as convergent power series. (This is true for almost all the functions encountered in pre-calculus and calculus.)

Such a power series representation is called a **Taylor series**.
When $c = 0$, the Taylor series is also called the **Maclaurin series**.

One power series representation we have already studied:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Whenever a function has a power series (Taylor or Maclaurin), the Derivative-Coefficient Identity may be applied to *calculate the coefficients* of that series.

Conversely, sometimes a series can be interpreted as an *evaluated power series* coming from $x = c$ for some c. If the closed form function format can be obtained for this power series, the *total sum of the original series may be discovered* by putting $x = c$ in the argument of the function.

02 Illustration

Example - Maclaurin series of e^x

What is the Maclaurin series of $f(x) = e^x$?

Solution

Using $\frac{d}{dx}e^x = e^x$ repeatedly, we see that $f^{(n)}(x) = e^x$ for all n.

So $f^{(n)}(0) = e^0 = 1$ for all n. Therefore $a_n = \frac{1}{n!}$ for all n by the Derivative-Coefficient Identity:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \; = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example - Maclaurin series of $\cos x$

Find the Maclaurin series representation of $\cos x$.

Solution

Use the Derivative-Coefficient Identity to solve for the coefficients:

$$a_n \; = \; \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/24$
5	$-\sin x$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By studying this pattern, we find the series:

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Maclaurin series from other Maclaurin series

(a) Find the Maclaurin series of $\sin x$ using the Maclaurin series of $\cos x$.

(b) Find the Maclaurin series of $f(x) = x^2 e^{-5x}$ using the Maclaurin series of e^x.

(c) Using (b), find the value of $f^{(22)}(0)$.

Solution

(a)

Remember that $\frac{d}{dx}\cos x = -\sin x$. Let us differentiate the cosine series by terms:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \overset{\frac{d}{dx}}{\ggg} \quad 0 - 2\frac{x^1}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$

$$\ggg \quad -\frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

Take negative to get:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(b)

$$e^u \;=\; 1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

Set $u = -5x$:

$$e^{-5x} \;=\; 1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{n!} x^n$$

Multiply all terms by x^2:

$$x^2e^{-5x} \quad \gg\gg \quad x^2\left(1+\frac{(-5x)}{1!}+\frac{(-5x)^2}{2!}+\frac{(-5x)^3}{3!}+\cdots\right)$$

$$\gg\gg \quad x^2-5x^3+\frac{25}{2}x^4-\frac{125}{3!}x^5+\cdots$$

$$\gg\gg \quad \sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}\,x^{n+2}$$

(c)

For any series:

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

we have:

$$f^{(n)}(0)=n!\cdot a_n$$

We can use this to compute a_{22}. From the series formula:

$$\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}\,x^{n+2}$$

we see that:

$$a_{n+2}=(-1)^n\frac{5^n}{n!}$$

⚠ Power, NOT term number

The coefficient with a_{n+2} corresponds to the term having x^{n+2}, *not necessarily* the $(n+2)^{\text{th}}$ term of the series.

Therefore:

$$a_{22} \;=\; (-1)^{20}\frac{5^{20}}{20!} \quad \gg\gg \quad 5^{20}\frac{1}{20!}$$

$$f^{(22)}(0) \;=\; 22!\cdot a_{22} \quad \gg\gg \quad 5^{20}\cdot\frac{22!}{20!} \quad \gg\gg \quad 5^{20}\cdot 22\cdot 21$$

Computing a Taylor series

Find the first five terms of the Taylor series of $f(x)=\sqrt{x+1}$ centered at $c=3$.

Solution

A Taylor series is just a Maclaurin series centered at a nonzero number.

General format of a Taylor series:

$$f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\cdots$$

The coefficients satisfy $a_n = \dfrac{f^{(n)}(c)}{n!}$.

Find the coefficients by computing the derivatives and evaluating at $x = 3$:

$$\begin{aligned} f(x) &= (x+1)^{1/2}, & f(3) &= 2 \\ f'(x) &= \frac{1}{2}(x+1)^{-1/2}, & f'(3) &= \frac{1}{4} \\ f''(x) &= -\frac{1}{4}(x+1)^{-3/2}, & f''(3) &= -\frac{1}{32} \\ f'''(x) &= \frac{3}{8}(x+1)^{-5/2}, & f'''(3) &= \frac{3}{256} \\ f^{(4)}(x) &= -\frac{15}{16}(x+1)^{-7/2}, & f^{(4)}(3) &= -\frac{15}{2048} \end{aligned}$$

The first terms of the series:

$$f(x) \; = \; \sqrt{x+1}$$

$$= \; 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \frac{5}{16,384}(x-3)^4 + \cdots$$

03 Theory

⚠ Study these!

- Memorize all of these series!
- Recognize all of these series!
- Recognize all of these summation formulas!

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots &&= \sum_{n=0}^{\infty} x^n, \quad R = 1, \quad \text{interval: } (-1, 1) \\ \ln(1-x) &= -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \cdots &&= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \quad R = 1, \quad \text{interval: } [-1, 1) \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad R = 1, \quad \text{interval: } [-1, 1] \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots &&= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = \infty \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad R = \infty \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad R = \infty \end{aligned}$$

Applications of Taylor series

Videos

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- Approximating with Maclaurin polynomials: $f(x) = \ln(1-x)$ to find $\ln(1.1)$

- Approximating with Taylor polynomials: $f(x) = \frac{1}{x+1}$ at $x = 1$ to find $1/2.1$

04 Theory reminder

Linear approximation is the technique of approximating a specific value of a function, say $f(x_1)$, at a point x_1 that is close to another point x_0 where we *know* the exact value $f(x_0)$. We write Δx for $x_1 - x_0$, and $y_0 = f(x_0)$, and $y_1 = f(x_1)$. Then we write $dy = f'(x_0) \cdot \Delta x$ and use the fact that:

$$y_1 \approx y_0 + dy = y_0 + f'(x_0) \cdot \Delta x$$

Computing a linear approximation

For example, to approximate the value of $\sqrt{4.01}$, set $f(x) = \sqrt{x}$, set $x_0 = 4$ and $y_0 = 2$, and set $x_1 = 4.01$ so $\Delta x = 0.01$.

Then compute: $f'(x) = \frac{1}{2\sqrt{x}}$
So $f'(x_0) = 1/4$.

Finally:

$$y_1 \approx y_0 + f'(x_0) \cdot \Delta x \qquad \ggg \qquad y_1 \approx 2 + \frac{1}{4} \cdot 0.01 = 2.0025$$

Now recall the **linearization** of a function, which is itself another function:

Given a function $f(x)$, the linearization $L(x)$ at the basepoint $x = c$ is:

$$L(x) = f(c) + f'(c)(x - c)$$

The graph of this linearization $L(x)$ is the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$.

The linearization $L(x)$ may be used as a replacement for $f(x)$ for values of x near c. The closer x is to c, the more accurate the approximation $L(x)$ is for $f(x)$.

Computing a linearization

We set $f(x) = \sqrt{x}$, and we let $c = 4$.

We compute $f(c) = 2$, and $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(c) = \frac{1}{4}$.

Plug everything in to find $L(x)$:

$$L(x) = f(c) + f'(c)(x - c) \quad \ggg \quad L(x) = 2 + \frac{1}{4}(x - 4)$$

Now approximate $f(4.01) \approx L(4.01)$:

$$L(4.01) = 2 + \frac{1}{4}(4.01 - 4) = 2.0025$$

05 Theory

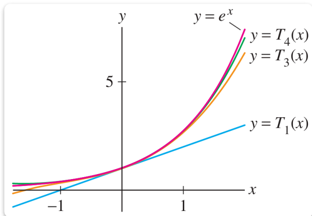
Taylor polynomials

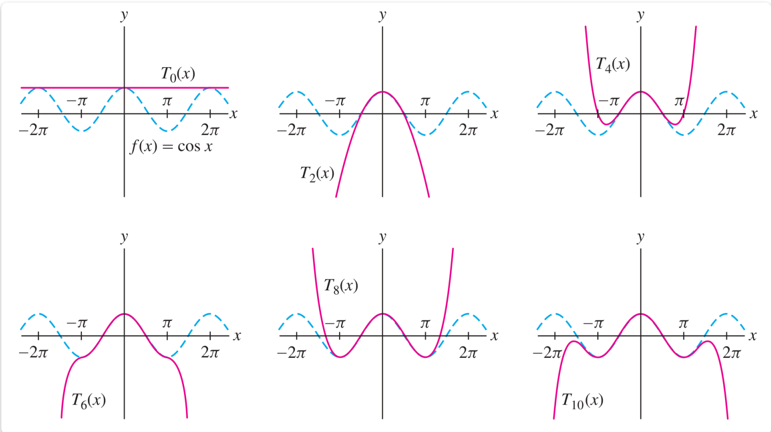
The **Taylor polynomials** $T_n(x)$ of a function $f(x)$ are the partial sums of the Taylor series of $f(x)$:

$$\begin{aligned} T_N(x) &= \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \end{aligned}$$

These polynomials are *generalizations of linearization*.
Specifically, $f(c) = T_0(x)$, and $L(x) = T_1(x)$.

The Taylor series $T_n(x)$ is a better approximation of $f(x)$ than $T_i(x)$ for any $i < n$.





Facts about Taylor series

The series $T_n(x)$ has the same derivatives as $f(x)$ at the point $x = c$. This fact can be verified by visual inspection of the series: apply the power rule and chain rule, then plug in $x = c$ and all factors left with $(x - c)$ will become zero.

The difference $f(x) - T_n(x)$ vanishes to order n at $x = c$:

$$\begin{aligned} f(x) - T_n(x) &= \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + \cdots \\ &= (x-c)^n \left(\frac{f^{(n)}(c)}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c) + \cdots \right) \end{aligned}$$

The factor $(x-c)^n$ drives the whole function to zero with order n as $x \to c$.

If we only considered orders up to n, we might say that $f(x)$ and $T_n(x)$ are the same near c.

06 Illustration

☰ Taylor polynomial approximations

Let $f(x) = \sin x$ and let $T_n(x)$ be the Taylor polynomials expanded around $c = 0$.

By considering the alternating series error bound, find the first n for which $T_n(0.02)$ must have error less than 10^{-6}.

Solution

Write the Maclaurin series of $\sin x$ because we are expanding around $c = 0$:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This series is alternating, so the AST error bound formula applies ("Next Term Bound"):

$$|E_n| \le a_{n+1}$$

Find smallest n such that $a_{n+1} \le 10^{-6}$, and then we know:

$$|E_n| \le a_{n+1} \le 10^{-6} \quad \gg\!\gg \quad |E_n| \le 10^{-6}$$

Plug $x = 0.02$ in the series for $\sin x$:

$$a_{2n+1} \;=\; \frac{(0.02)^{2n+1}}{(2n+1)!}$$

Solve for the first time $a_{2n+1} \le 10^{-6}$ by listing the values:

$$\frac{0.02^1}{1!} = 0.02, \qquad \frac{0.02^3}{3!} \approx 1.33 \times 10^{-6},$$

$$\frac{0.02^5}{5!} \approx 2.67 \times 10^{-11}, \quad \ldots$$

The first time a_{2n+1} is below 10^{-6} happens when $2n + 1 = 5$.

This is NOT the same n as in T_n. That n is the highest power of x allowed.

The sum of prior terms is $T_4(0.02)$.

Since $T_4(x) = T_3(x)$ because there is no x^4 term, the final answer is $n = 3$.

☰ Taylor polynomials to approximate a definite integral

Approximate $\displaystyle\int_0^{0.3} e^{-x^2}\,dx$ using a Taylor polynomial with an error no greater than 10^{-5}.

Solution

Plug $u = -x^2$ into the series of e^u:

$$e^u \ = \ 1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots$$

$$\gg\gg \quad e^{-x^2} \ = \ 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

Antiderivative by terms:

$$\int 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \, dx$$

$$\gg\gg \quad C + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Plug in bounds for definite integral:

$$\int_0^{0.3} e^{-x^2}\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \Big|_0^{0.3}$$

$$\gg\gg \qquad 0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} - \frac{0.3^7}{7!} + \cdots$$

Notice alternating series, apply error bound formula “Next Term Bound”:

$$\frac{0.3^3}{3!} \approx 0.0045, \qquad \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}, \qquad \frac{0.3^7}{7!} \approx 4.34 \times 10^{-8}$$

So we can guarantee an error less than 4.34×10^{-5} by summing the first terms through $\frac{0.3^5}{5!}$:

$$0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} \quad \gg\gg \quad \approx 0.291243$$