W10 Reg OS 1(b)

$$\sum_{n=1}^{\infty} \frac{x^n}{3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)}$$

e.g. $n=4 \rightsquigarrow 4n-1 = 15$

denom $\rightsquigarrow 3\cdot 7\cdot 11\cdot 15$

e.g. $n=5 \rightsquigarrow 4n-1 = 19$

denom $\rightsquigarrow 3\cdot 7\cdot 11\cdot 15\cdot 19$

Plug in $n+1$ for $n$:

$\rightsquigarrow 3\cdot 7\cdot 11\cdot \ldots \cdot (\underbrace{4(n+1)-1}_{\to\, 4n+3})$

$3\cdot 7\cdot 11\cdot \ldots \cdot (4n+3) = 3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)(4n+3)$

$$\left| \frac{a_{n+1} = \dfrac{x^{n+1}}{3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)(4n+3)}}{a_n = \dfrac{x^n}{3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)}} \right| \rightsquigarrow \frac{\cancel{3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)}}{\cancel{3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1)}(4n+3)} |x|$$

$$\rightsquigarrow \frac{1}{4n+3}|x| \xrightarrow{n\to\infty} 0$$

So $R = \infty$, $I = (-\infty, +\infty)$.

# Power Series As Functions

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots = \sum_{n=0}^{\infty} b_n x^n$$

when converge

**Scaling:** $(cf)(x) = \sum_{n=0}^{\infty} (c a_n) x^n$

**Adding:** $(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

**Multiplication:** $(fg)(x) = \sum_{n=0}^{\infty} \Big( \sum_{i+j=n} a_i b_j \Big) x^n$ $\quad = \sum_{i=0}^{n} a_i b_{n-i}$

i.e. $(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$

**Differentiation:** $\frac{df}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$

**Integration:** $\int f\,dx = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

$$\int_a^b f\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \Big|_a^b = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \left( b^{n+1} - a^{n+1} \right)$$

↳ **Convergence:** Suppose $f$ has radius of conv. $R$.

Then $f'(x)$ and $\int f\,dx$ also have radius of conv. $R$.

⚠ Endpoint convergence facts may change!

# Manipulating Geometric Series

– segue to Taylor Series –

**Examples:** Write as power series:

(a) $\frac{1}{1+x}$ (b) $\frac{1}{1+x^2}$ (c) $\frac{x^3}{x+2}$ (d) $\frac{3x}{2-5x}$

**Solution:** Method: use $\frac{1}{1-x} \overset{|x|<1}{=} 1+x+x^2+x^3+\ldots = \sum_{n=0}^{\infty} x^n$

from $\frac{a}{1-r} = \sum ar^n$
$r \rightsquigarrow x$
$a \rightsquigarrow 1$

(a) $\frac{1}{1+x} = \frac{1}{1-(-x)} \rightsquigarrow \sum_{n=0}^{\infty} (-x)^n$

$\rightsquigarrow \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$ ⚠ write it in standard form

(b) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \rightsquigarrow \sum_{n=0}^{\infty} (-x^2)^n \rightsquigarrow \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n}}$

$= 1 - x^2 + x^4 - x^6 + x^8 - \ldots$

(c) $\frac{x^3}{x+2} = x^3\left(\frac{1}{2+x}\right) = \frac{x^3}{2}\left(\frac{1}{1-(-x/2)}\right)$

$\rightsquigarrow \frac{x^3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \rightsquigarrow \sum_{n=0}^{\infty} \frac{x^3}{2}(-1)^n \frac{1}{2^n} x^n$

$\rightsquigarrow \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} x^{n+3}}$

(d) $\frac{3x}{2-5x} = 3x\left(\frac{1}{2-5x}\right) = \frac{3x}{2}\left(\frac{1}{1-\frac{5}{2}x}\right)$

$\rightsquigarrow \frac{3x}{2}\sum_{n=0}^{\infty}\left(\frac{5}{2}x\right)^n \rightsquigarrow \sum_{n=0}^{\infty} \frac{3x}{2}\left(\frac{5}{2}\right)^n x^n \rightsquigarrow \boxed{\sum_{n=0}^{\infty} 3\frac{5^n}{2^{n+1}} x^{n+1}}$

**Example:** $\ln(1+x) = ?$

Notice: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

$\int$

So: $\ln(1+x) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$ for some $C$.

Plug: $x=0 \rightsquigarrow \ln(1) = C+0 \rightsquigarrow 0 = C.$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}}$$

Cool consequence: $\ln(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$

---

**Example:** (a) Evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

Reindex: $n \rightsquigarrow n+1$: $\sum_{n+1=1}^{\infty}(-1)^{n+1-1}\frac{1}{n+1}$

$\rightsquigarrow \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1} = \;=\boxed{\ln(2)}$

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**Example:** (a) $\tan^{-1}(x)$ (b) $\int\frac{dx}{1+x^4}$

Solution: (a) $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$\rightsquigarrow \tan^{-1}(x) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$

$\rightsquigarrow$ @ $x=0$: $0 = C+0 \rightsquigarrow C=0.$

(b) $\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} \rightsquigarrow \sum_{n=0}^{\infty}(-x^4)^n \rightsquigarrow \sum_{n=0}^{\infty}(-1)^n x^{4n}$

$\rightsquigarrow \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}x^{4n+1}}$

# Taylor Series

Derivative-Coefficient Identity:

$$f^{(n)}(0) = n!\, a_n$$

$$\text{when } f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

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**Example:** Find series of $f(x) = e^x$.

Notice: $\frac{d}{dx} e^x = e^x$ so $\frac{d^n}{dx^n} e^x = e^x$

So $f^{(n)}(x) = e^x$, $f^{(n)}(0) = e^0 = 1$

Through D-C Id.: $f^{(n)}(0) = n!\, a_n$

$\rightsquigarrow 1 = n!\, a_n$

$\rightsquigarrow a_n = \frac{1}{n!}$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n} = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

---

**Example:** Find series for $f(x) = \cos(x)$.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $a_n$ |
|---|---|---|---|
| 0 | $\cos x$ | 1 | 1 |
| 1 | $-\sin x$ | 0 | 0 |
| 2 | $-\cos x$ | $-1$ | $-1/2!$ |
| 3 | $+\sin x$ | 0 | 0 |
| 4 | $+\cos x$ | 1 | $1/4!$ |
| 5 | $-\sin x$ | 0 | 0 |
| 6 | $-\cos x$ | $-1$ | $-1/6!$ |
| $\vdots$ | | | |

$\rightsquigarrow \cos x = \sum_{n=0}^{\infty} a_n x^n$ $\rightsquigarrow$ reindex: $2n \rightsquigarrow n$

$\rightsquigarrow \cos x = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 - 0 - \frac{1}{6!}x^6 + \dots$

$$\rightsquigarrow \boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}}$$

<u>Example:</u> (a) Series of $\sin x$
(b) Series of $f(x) = x^2 e^{-5x}$
(c) Value of $f^{(22)}(0)$

<u>Solution:</u>

(a) Recall $\sin x = -\frac{d}{dx}\cos x$

$$= -\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\rightsquigarrow -\left(\sum_{n=1}^{\infty}(-1)^n(2n)\frac{x^{2n-1}}{(2n)!}\right)$$

$$\rightsquigarrow \boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}}$$

ALSO $\boxed{= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}}$

(b) $x^2 e^{-5x}$

$$e^u = \sum_{n=0}^{\infty}\frac{u^n}{n!} \quad @\ u = -5x$$

$$\rightsquigarrow e^{-5x} = \sum_{n=0}^{\infty}\frac{(-5x)^n}{n!}$$

$$\rightsquigarrow \sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}x^n$$

So $x^2 e^{-5x} = x^2\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}x^n$

$$\rightsquigarrow \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}x^{n+2}}$$

(c) $f^{(22)}(0)$ when $f(x) = x^2 e^{-5x}$.

Use: $\underline{f^{(22)}(0) = 22!\, a_{22}}$

meaning of $a_{22}$ from: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$$= \sum_{n=0}^{\infty} a_n x^n$$

i.e. $\underline{\text{coef. of } x^{22}}$

We have: $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{n!} x^{n+2}$

So $x^{22} = x^{n+2} \rightsquigarrow n = 20$

$n \neq n \dots$

$a_{22} = (-1)^{20} \frac{5^{20}}{20!}$, $\boxed{f^{(22)}(0) = 22!\, \frac{5^{20}}{20!}}$

# Taylor @ $x = c$

Can represent a function as Taylor Series centered at other $x = c$:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Get new Der.-Coef. Identity:

$$f^{(n)}(c) = n!\, a_n$$

"Maclaurin Series" = "Taylor Series" with $c = 0$

$= (x+1)^{1/2}$

**Example:** $f(x) = \sqrt{x+1}$, Taylor @ $x = 3$.

Solution:

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(c)$ | $a_n$ |
|---|---|---|---|
| 0 | $(x+1)^{1/2}$ | $2$ | $2$ |
| 1 | $\frac{1}{2}(x+1)^{-1/2}$ | $1/4$ | $1/4$ |
| 2 | $-\frac{1}{4}(x+1)^{-3/2}$ | $-\frac{1}{32}$ | $-1/64$ |
| 3 | $\frac{3}{8}(x+1)^{-5/2}$ | $3/256$ | $1/512$ |
| 4 | $-\frac{15}{16}(x+1)^{-7/2}$ | $-\frac{15}{2048}$ | $-\frac{5}{16{,}384}$ |
| 5 | | | |
| $\vdots$ | | | |

$$\leadsto \sqrt{x+1} = 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 + \dots$$

Can you find general term?

To write as summation?

**Example:** Evaluate $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2^{4n}} \cdot \frac{1}{(2n)!}$

**Solution:** Look: $\dfrac{\pi^{2n}}{2^{4n}} = \left(\dfrac{\pi}{4}\right)^{2n}$

$$\rightsquigarrow \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$@\ x = \frac{\pi}{4}$

This is ... $\cos x$

$$\boxed{\text{Answer} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}}$$

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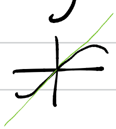
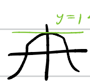
(!) Memory Aid:

$\sin x$ : 

$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots$

$\cos x$ : 

$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

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## Applications: Taylor Polynomials, Taylor Approximations

**Taylor Polynomials:**

$$T_N(x) = a_0 + a_1(x-c) + \dots + a_N(x-c)^N$$

$$= \sum_{n=0}^{N} a_n x^n$$

$$a_n = \frac{f^{(n)}(c)}{n!}$$

**Taylor Approximation:** e.g. $T_4(0.002)$ $@\ c = 0$

$\approx f(0.002)$