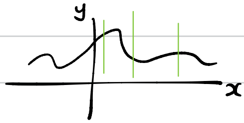
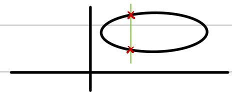
4/1

Parametric Curves

Recall: graph of function $y = f(x) \rightsquigarrow (x, f(x))$

Therefore $x \mapsto y$





So: asymmetric treatment of x vs. y.

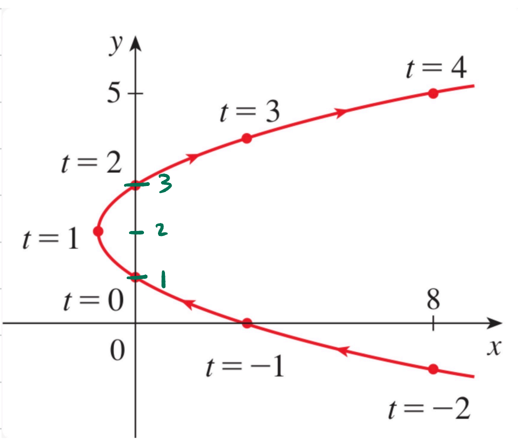
Now parametric curves: treat x, y same way

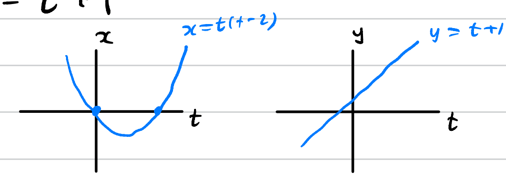
$$t \xmapsto{f} x = f(t) \qquad \text{OR} \qquad (x(t), y(t))$$
$$t \xmapsto[g]{} y = g(t)$$

"Moving point particle."

Image of $(x, y) = (f(t), g(t))$ is set of points traversed

Example: $x = t^2 - 2t$, $y = t + 1$

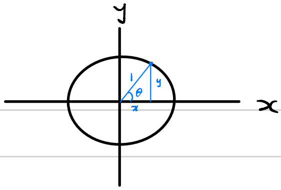




$$x = t^2 - 2t, \quad y = t + 1$$
$$\rightsquigarrow t = y - 1$$
$$\rightsquigarrow x = (y-1)^2 - 2(y-1)$$
$$= y^2 - 2y + 1 - 2y + 2$$
$$= y^2 - 4y + 3$$
$$x = (y-2)^2 - 1$$

$\rightsquigarrow$ parabola, opens $+x$

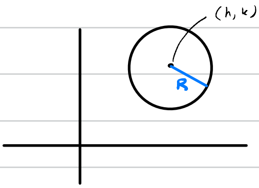
Example: Unit Circle



$$x^2+y^2=1 \qquad \cos^2\theta+\sin^2\theta=1$$

$$\rightsquigarrow \quad \left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \Rightarrow x^2+y^2=1$$

General Circle



$$(x-h)^2+(y-k)^2 = R^2$$

$$x = h + R\cos t$$
$$y = k + R\sin t$$

See: $(h+R\cos t - h)^2 + (k+R\sin t - k)^2 \stackrel{?}{=} R^2$

$$\rightsquigarrow \quad R^2\cos^2 t + R^2\sin^2 t$$
$$\rightsquigarrow \quad R^2(\cos^2 t + \sin^2 t)$$
$$\rightsquigarrow \quad R^2 = \checkmark \qquad R^2$$

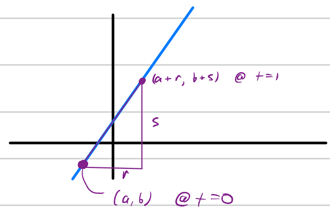
Ellipse

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

$$x = h + a\cos t$$
$$y = k + b\sin t$$

Example: Lines

$$x = a + rt \qquad t \in (-\infty, +\infty)$$
$$y = b + st$$



Start w/ $(x,y) = (a+rt,\ b+st)$ $\rightsquigarrow$ $y = mx + b$:

slope $= \frac{s}{r}$

y-intercept $= b - \frac{s}{r}a$

$\rightsquigarrow t = \frac{x-a}{r}$

$\rightsquigarrow y = b + s\left(\frac{x-a}{r}\right)$

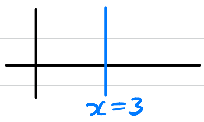
$\rightsquigarrow y = \left(\frac{s}{r}\right)x + \left(b - \frac{s}{r}a\right)$

Start w/ $y = mx + b$ $\rightsquigarrow$ a, r, b, s:

$y = mt + b, \quad x = t$ i.e. $a = 0,\ r = 1$

$b = b,\ s = m$

Start with $x = 3$



$x = 3 \qquad a = 3,\ r = 0$

$y = t \qquad b = 0,\ s = 1$

Could actually use any satisfying: $a = 3,\ r = 0$

$b = \text{any},\ s \neq 0$

Calculus with Parametric Curves

Given $x(t)$, $y(t)$. Slope:

$$\boxed{m = \frac{dy}{dx} = \frac{y'}{x'}}$$ where $x'(t) \neq 0$

Why? $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} \rightsquigarrow \dfrac{dy}{\cancel{dt}} \cdot \dfrac{\cancel{dt}}{dx} \rightsquigarrow \dfrac{dy}{dx}$

More rigorous: $\dfrac{d}{dx}(y(t)) = \dfrac{d}{dt}(y) \cdot \dfrac{dt}{dx}$ (chain rule)

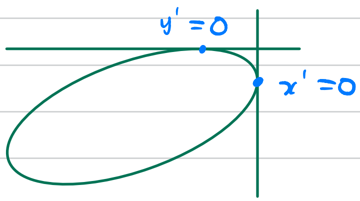
Recall: $\dfrac{d}{dy} f^{-1}(y) = \dfrac{1}{f'(x)}$ from implicit diff (inverse fcn rule)

so $\dfrac{dt}{dx} = \dfrac{1}{x'}$

$$\rightsquigarrow \frac{d}{dx} y = \frac{dy}{dt} \cdot \frac{1}{x'} = \frac{y'}{x'}$$

From the formula:

- Pure vertical motion when: $x' = 0$ & $y' \neq 0$
- Pure horizontal motion when: $y' = 0$ & $x' \neq 0$
- "Stationary point" when: BOTH $x' = 0$, $y' = 0$



Must verify both $x' = 0$, $y' \neq 0$ (e.g. for verticals)

Example: Circle $x = \cos t$, $y = \sin t$

Find points with vert. or horiz. tangent.

Solution: $x' = -\sin t$, $y' = \cos t$

Verticals: $x' = 0 \rightsquigarrow -\sin t = 0 \rightsquigarrow t = 0, \pi$

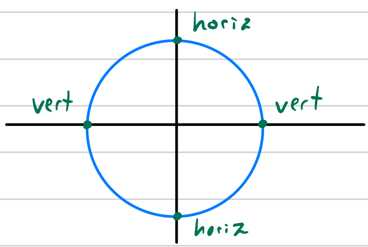
Check: $\left.\begin{array}{l} y'(0) = +1 \\ y'(\pi) = -1 \end{array}\right\} \neq 0$ ✓

Points: $(\cos t, \sin t)\Big|_{t=0,\pi} \rightsquigarrow \underline{(1,0)}, \underline{(-1,0)}$

Horizontals: $y' = 0 \rightsquigarrow \cos t = 0 \rightsquigarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

Check: $\left.\begin{array}{l} x'(\frac{\pi}{2}) = -1 \\ x'(\frac{3\pi}{2}) = +1 \end{array}\right\} \neq 0$ ✓

Points: $(\cos t, \sin t)\Big|_{t=\frac{\pi}{2}, \frac{3\pi}{2}} \rightsquigarrow \underline{(0,1)}, \underline{(0,-1)}$



Example: $x = t^2, \quad y = t^3$

Find where slope $= 5$.

Solution:

$$m = \frac{y'}{x'} = \frac{3t^2}{2t} \rightsquigarrow \frac{3}{2}t$$

Solve: $\frac{3}{2}t = 5 \rightsquigarrow t = \frac{10}{3}$

Point: plug in: $\left(\frac{100}{9}, \frac{1000}{27}\right) = (t^2, t^3)\Big|_{t=\frac{10}{3}}$

Concavity

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{1}{x'}}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \rightsquigarrow \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

chain rule $\rightsquigarrow \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{dt}{dx}$ $\overset{\text{inverse fcn again}}{\rightsquigarrow} \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{1}{x'}$

Example: Find tangent line to cycloid $(4t - 4\sin t,\ 4 - 4\cos t)$ $t = \pi/4$

Solution: $x' = 4 - 4\cos t$, $y' = 4\sin t$

Slope: $m = \frac{dy}{dx} = \frac{y'}{x'} = \frac{4\sin t}{4 - 4\cos t}$

@ $t = \pi/4$: $\rightsquigarrow \frac{4\sin \pi/4}{4 - 4\cos\frac{\pi}{4}} \rightsquigarrow \frac{2\sqrt{2}}{4 - 2\sqrt{2}} = m$

Point-Slope Form:

@ $t = \frac{\pi}{4}$ point: $\left(4\cdot\frac{\pi}{4} - 4\sin\frac{\pi}{4},\ 4 - 4\cos\frac{\pi}{4}\right)$

$\rightsquigarrow \left(\pi - 2\sqrt{2},\ 4 - 2\sqrt{2}\right)$

$$y - (4 - 2\sqrt{2}) = \left(\frac{2\sqrt{2}}{4 - 2\sqrt{2}}\right)\left(x - (\pi - 2\sqrt{2})\right)$$

Parametric Line: $x = a + rt,\ y = b + st$

$$\rightsquigarrow \boxed{\begin{cases} x = \pi - 2\sqrt{2} + (4 - 2\sqrt{2})t \\ y = 4 - 2\sqrt{2} + (2\sqrt{2})t \end{cases}}$$