W13 - Examples

Parametric lines

(1) Parametric coordinate functions for a line:

$$x = a + rt, \qquad y = b + st, \qquad t \in (-\infty, +\infty)$$

Compare this to the graph of linear function:

$$y = mx + b \qquad \text{some } m, b$$

Vertical lines cannot be described as the graph of a function. We must use $x = a$.

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(2) Parametric lines can describe all lines equally well, including horizontal and vertical lines.

A vertical line $x = a$ is achieved by setting $s = 0$ and $r \neq 0$.

A horizontal line $y = b$ is achieved by setting $r = 0$ and $s \neq 0$.

A non-vertical line $y = mx + b$ may be achieved by setting $s = m$ and $r = 1$, and $a = 0$.

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(3) Assuming that $r \neq 0$, the parametric coordinate functions describe a line satisfying:

$$y = b + s\left(\frac{x-a}{r}\right)$$

$$\gg\gg \quad y = \frac{s}{r} \cdot x + \left(b - \frac{s}{r} \cdot a\right)$$

and therefore the slope is $m = \frac{s}{r}$ and the y-intercept is $b - \frac{s}{r} \cdot a$.

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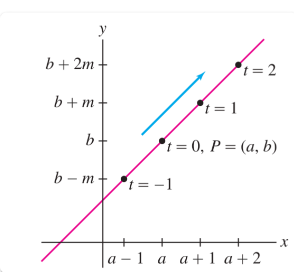
(4) The point-slope construction of a line has a parametric analogue:

point-slope line:

$$y - a = m(x - b) \qquad\qquad (x, y) = (a + t,\, b + mt)$$



Parametric ellipses

The general equation of an ellipse centered at (h,k) with half-axes a and b is:

$$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$$

This equation represents a *stretched unit circle*:

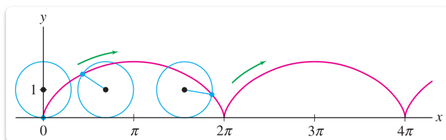
- by a in the x-axis
- by b in the y-axis

Parametric coordinate functions for the general ellipse:

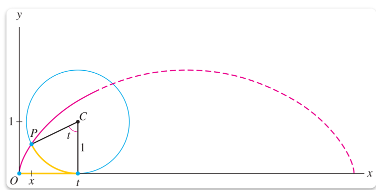
$$x=h+a\cos t,\qquad y=k+b\sin t,\qquad t\in[0,2\pi)$$

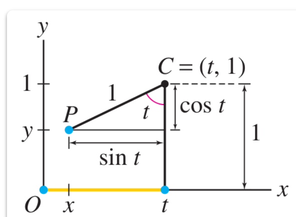
Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:





The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t,1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t-\sin t$. The y coordinate is $1-\cos t$.

So the coordinates of the point $P=(x,y)$ are given parametrically by:

$$x=t-\sin t,\qquad y=1-\cos t,\qquad t>0$$

If the circle has another radius, say R, then the parametric formulas change to:

$$x=Rt-R\sin t,\qquad y=R-R\cos t,\qquad t>0$$

Tangent to a cycloid

Find the tangent line (described parametrically) to the cycloid $(4t-4\sin t,\ 4-4\cos t)$ when $t=\pi/4$.

Solution

(1) Compute x' and y'.

Find $x'(t)$ and $y'(t)$:

$$x(t) = 4t - 4\sin t \quad \gg\gg \quad x'(t) = 4 - 4\cos t$$

$$y(t) = 4 - 4\cos t \quad \gg\gg \quad y'(t) = 4\sin t$$

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(2) Plug in $t = \pi/4$:

$$x'(\pi/4) \quad \gg\gg \quad 4 - 4\cos(\pi/4) \quad \gg\gg \quad 4 - 2\sqrt{2}$$

$$y'(\pi/4) \quad \gg\gg \quad 4\sin(\pi/4) \quad \gg\gg \quad 2\sqrt{2}$$

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(3) Apply formula: $\dfrac{dy}{dx} = \dfrac{y'}{x'}$:

Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \qquad \gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

$$\gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}}$$

$$\gg\gg \qquad \frac{8\sqrt{2} + 8}{16 - 8} \quad \gg\gg \quad \sqrt{2} + 1$$

So:

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2} + 1$$

This is the slope m for our line.

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(4) Need the point P for our line. Find (x, y) at $t = \pi/4$.

Plug $t = \pi/4$ into parametric formulas:

$$\big(x(t),\, y(t)\big)\Big|_{t=\pi/4} \quad \gg\gg \quad \Big(4\frac{\pi}{4} - 4\sin(\pi/4),\ 4 - 4\cos(\pi/4)\Big)$$

$$\gg\gg \quad \Big(\pi - 2\sqrt{2}, 4 - 2\sqrt{2}\Big)$$

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(5) Point-slope formulation of tangent line:

$$x = a + t, \quad y = b + mt$$

Inserting data:

$$x = (\pi - 2\sqrt{2}) + t, \qquad y = (4 - 2\sqrt{2}) + (\sqrt{2} + 1)t$$

Vertical and horizontal tangents of the circle

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

Solution

(1) For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$x'(t) = -\sin t,$$

$$x'(t) = 0 \qquad \gg\!\gg \qquad -\sin t = 0$$

$$\gg\!\gg \qquad t = 0,\ \pi$$

The moving point is at $(1, 0)$ when $t = 0$, and at $(-1, 0)$ when $t = \pi$.

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(2) For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$y'(t) = \cos t,$$

$$y'(t) = 0 \qquad \gg\!\gg \qquad \cos t = 0$$

$$\gg\!\gg \qquad t = \frac{\pi}{2},\ \frac{3\pi}{2}$$

The moving point is at $(0, 1)$ when $t = \pi/2$, and at $(0, -1)$ when $t = 3\pi/2$.

Finding the point with specified slope

Consider the parametric curve given by $(x, y) = (t^2,\ t^3)$. Find the point where the slope of the tangent line to this curve equals 5.

Solution

(1) Compute the derivatives:

$$x'(t) \ = \ 2t, \qquad y'(t) \ = \ 3t^2$$

Therefore the slope of the tangent line, in terms of t:

$$m = \frac{dy}{dx} \ = \ \frac{y'(t)}{x'(t)}$$

$$\gg\!\gg \quad \frac{3t^2}{2t} \quad \gg\!\gg \quad \frac{3}{2}t$$

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(2) Set up equation:

$$m \ = \ 5$$

$$\frac{3}{2}t \ = \ 5$$

$$\gg\!\gg \quad t = \frac{10}{3}$$

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(3) Find the point:

$$(x,y)\Big|_{t=10/3} \quad \gg\!\gg \quad \left(\frac{100}{9},\ \frac{1000}{27}\right)$$

Perimeter of a circle

(1) The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 \;=\; (-R\sin t)^2 + (R\cos t)^2$$

$$\gg\!\gg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \gg\!\gg \qquad R^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = R\, dt$$

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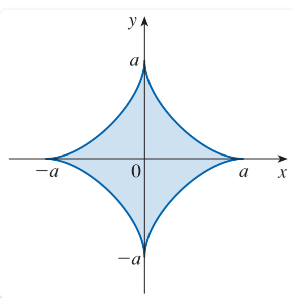
(2) Integrate around the circle:

$$\text{Perimeter} \;=\; \int_0^{2\pi} ds \quad \gg\!\gg \quad \int_0^{2\pi} R\, dt$$

$$\gg\!\gg \quad Rt\Big|_0^{2\pi} = 2\pi R$$

Perimeter of an asteroid

Find the perimeter length of the ‘asteroid’ given parametrically by $(x,y) = \big(a\cos^3\theta,\ a\sin^3\theta\big)$ for $a = 2$.



Solution

(1) Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$\big(x', y'\big) = \big(3a\cos^2\theta\sin\theta,\ 3a\sin^2\theta\cos\theta\big)$$

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(2) Compute the infinitesimal arc element.

$$(x')^2 + (y')^2 \quad \gg\gg \quad 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$\gg\gg \quad 9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$\gg\gg \quad 9a^2\sin^2\theta\cos^2\theta$$

Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + y')^2}\,d\theta$$

$$\gg\gg \quad \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta$$

$$\gg\gg \quad ds = 3a\big|\sin\theta\cos\theta\big|\,d\theta$$

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(3) Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a\sin\theta\cos\theta$ is *positive*. So we can drop the absolute value and integrate directly.

> ⚠ **Absolute values matter!**
>
> If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a\sin\theta\cos\theta$ really does change sign.
>
> To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

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(4) Integrate the arc element:

$$\int_0^{\pi/2} ds \quad \gg\gg \quad \int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta$$

$$\gg\gg \quad 3a\int_{u=0}^{1} u\,du \qquad (u = \sin\theta)$$

$$\gg\gg \quad 3a\frac{u^2}{2}\bigg|_0^1 \quad \gg\gg \quad \frac{3a}{2}$$

Finally, multiply by 4 to get the total perimeter: $L = 6a$