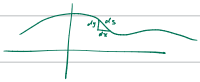
Parametric Arclength

$$L = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2}$$



$y = f(x)$

$\sqrt{1+f'^2}\,dx$

$\rightsquigarrow \sqrt{dx^2 + (f'dx)^2}$

$f'dx = \frac{dy}{dx}dx = dy$

$$\rightsquigarrow \int_a^b \sqrt{x'^2 + y'^2}\, dt$$

$\sqrt{(x'dt)^2 + (y'dt)^2}$

$= \sqrt{dx^2 + dy^2}$

Example: Circle $(R\cos t, R\sin t)$. Arclength?

Solution:

$$L = \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

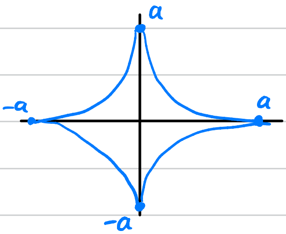
$R^2(\sin^2 + \cos^2 t) = R^2$

$$\rightsquigarrow \int_0^{2\pi} \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt$$

$$\rightsquigarrow \int_0^{2\pi} R\, dt \rightsquigarrow Rt\Big|_0^{2\pi} \rightsquigarrow \boxed{2\pi R}$$

Example: Perimeter of "asteroid":

$(x, y) = (a\cos^3\theta, a\sin^3\theta)$ $\quad a = 2$



Solution: $L = 4\int_0^{\pi/2} ds$

$$ds = \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$\rightsquigarrow 3a\sqrt{\cos^4\sin^2 + \sin^4\cos^2}\, d\theta$$

$$\rightsquigarrow 3a\sqrt{\sin^2\cos^2(\cos^2+\sin^2)}\, d\theta \rightsquigarrow 3a\,|\sin\theta\cos\theta|\, d\theta$$

So $$L = 4\int_0^{\pi/2} 3a\,|\sin\theta\cos\theta|\,d\theta$$

For $0 \le \theta \le \pi/2$ have $\begin{matrix}\sin\theta \geq 0 \\ \cos\theta \geq 0\end{matrix} \rightsquigarrow \begin{matrix}|\sin\theta\cos\theta| \\ = \sin\theta\cos\theta\end{matrix}$

So $$L = 4\int_0^{\pi/2} 3a \sin\theta\cos\theta\,d\theta$$

$u = \sin\theta$, $du = \cos\theta\,d\theta$, $\begin{matrix}0 \to 0 \\ \pi/2 \to 1\end{matrix}$

$$\rightsquigarrow L = 4\int_0^1 3a\,u\,du \rightsquigarrow \boxed{6a}$$

Distance & Speed

$$s(t) = \int_{t_0}^{t} ds = \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\,du$$ "distance traveled function"

$$v(t) = s'(t) = \sqrt{x'^2 + y'^2}$$ "speed"

⚠ Distance traveled by $(x(t), y(t))$ point, $0 \le t \le t_0$
Odometer reading.

Might double count! (If reverses...)

Example: $(x, y) = \left(t, \frac{2}{3} t^{3/2}\right)$

(a) Speed fcn?

Assume now travel for 8 seconds, starting $t=0$.
(b) Find total distance traveled (c) Find total displacement.

Solution: (a) $x' = 1, \quad y' = t^{1/2}$

$$\leadsto \sqrt{x'^2 + y'^2} = \boxed{\sqrt{1+t}}$$

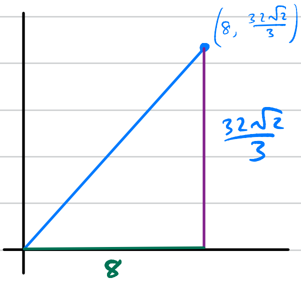
(b) $\overset{t_0=0}{s(8)} = \int_0^8 \sqrt{1+t}\, dt$ $\qquad$ $1+t = u$, $\quad dt = du$

$$\leadsto \int_1^9 \sqrt{u}\, du \leadsto \frac{2}{3} u^{3/2} \Big|_1^9$$

$$\leadsto \frac{2}{3}(27-1) \leadsto \boxed{\frac{52}{3}}$$

(c) @ $t=0$, $(x,y) = (0,0)$
@ $t=8$, $(x,y) = \left(8, \frac{32\sqrt{2}}{3}\right)$

Displacement = distance between these:

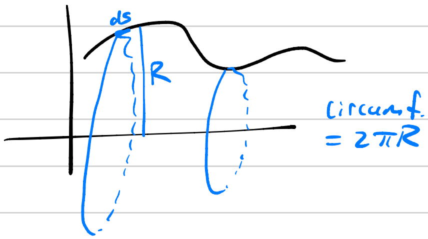


$$\sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2}$$

$$\leadsto \boxed{\frac{\sqrt{2624}}{3}}$$

Parametric Surface Area of Revolutions

$$A = \int_a^b 2\pi R\, ds \;\rightsquigarrow\; \int_a^b 2\pi R(t)\sqrt{x'^2 + y'^2}\, dt$$



$R(t) = y(t)$ rev. about x-axis

$R(t) = x(t)$ rev. about y-axis

Example: unit Sphere by revolving $(\cos t, \sin t)$
$0 \le t \le \pi$

$$A = \int_0^{\pi} 2\pi \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$\rightsquigarrow \int_0^{\pi} 2\pi \sin t\, dt \;\rightsquigarrow\; -2\pi \cos t \Big|_0^{\pi} \;\rightsquigarrow\; \boxed{4\pi}$$

Example: Set up integral for surface area:
$(x,y) = (t^3, t^2 - 1)$, $0 \le t \le 1$ about x-axis

Solution: $x'^2 + y'^2 \rightsquigarrow (3t^2)^2 + (2t)^2 \rightsquigarrow t^2(9t^2+4)$

$ds \rightsquigarrow t\sqrt{9t^2+4}$

$$\boxed{A = \int_0^{\pi} 2\pi (t^2 - 1)\, t\sqrt{9t^2+4}\; dt}$$

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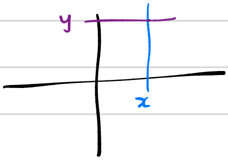
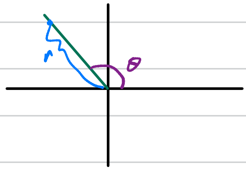
Polar Coordinates





Polar → Cartesian

$x = r\cos\theta$

$y = r\sin\theta$

No issues.

Cartesian → Polar

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

Issues...

1. θ from x, y domain issue...
2. $\frac{y}{x}$ requires $x \neq 0$
3. r forced positive

Say point $(x,y) = (-1,-1) \rightsquigarrow (r,\theta) = (\sqrt{2}, \frac{5\pi}{4})$

Say point $(x,y) = (1,1) \rightsquigarrow (r,\theta) = (\sqrt{2}, \frac{\pi}{4})$

$\tan^{-1}(-1/-1) = \tan^{-1}(1) = \frac{\pi}{4}$

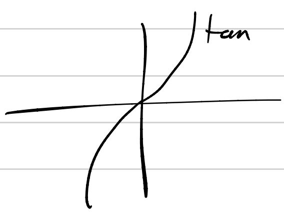
$\tan(\frac{5\pi}{4}) = 1 = \tan(\frac{\pi}{4})$

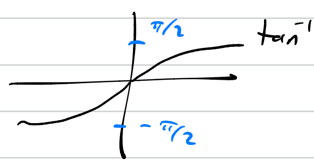
So... Quadrants II, III unsafe $\rightsquigarrow$ must correct

Quadrants I, IV safe

Quadrant II, III : $\theta = \tan^{-1}(y/x) \underline{+ \pi}$

Quadrant I, IV : $\theta = \tan^{-1}(y/x)$





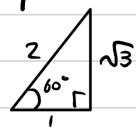
Example: $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Cartesian to Polar

Solution: $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ Quadrant II unsafe

$$r = \sqrt{(-1/2)^2 + (\sqrt{3}/2)^2} = \sqrt{1/4 + 3/4} = 1$$

$$\begin{aligned}\theta &= \tan^{-1}(y/x) + \pi \\ &= \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) + \pi \\ &= -\pi/3 + \pi \\ &= 2\pi/3 \quad \rightsquigarrow (r,\theta) = \boxed{(1, 2\pi/3)}\end{aligned}$$



$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ safe (Quad. IV)

$$r = \sqrt{2/4 + 2/4} = 1$$

$$\theta = \tan^{-1}(-1) = -\pi/4 \qquad (r,\theta) = (1, -\pi/4)$$
$$= \boxed{(1, 7\pi/4)}$$

Convert Equations

Use opposite conversion direction formulas.

Advice: preserve trig functions

$$r = \sin^2\theta \quad \rightsquigarrow \quad \sqrt{x^2+y^2} = \frac{y^2}{r^2} = \frac{y^2}{x^2+y^2}$$

$$\Updownarrow$$

$$\boxed{(x^2+y^2)^{3/2} = y^2}$$

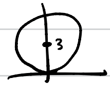
$r = \sqrt{x^2+y^2}$

$\sin\theta = y/r$

$\cos\theta = x/r$

$\tan\theta = y/x$

Example: $x^2 + (y-3)^2 = 9$



$x = r\cos\theta$
$y = r\sin\theta$

$$r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\leadsto r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\leadsto r^2(\cos^2 + \sin^2) = 6r\sin\theta$$

$$\leadsto r^2 = 6r\sin\theta$$

$$\leadsto \boxed{r = 6\sin\theta}$$