W14 - Examples

Speed, distance, displacement

The parametric curve $\left(t, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)

Compute *derivatives*:

$$(x', y') = \left(1, t^{1/2}\right)$$

Now compute the *speed*:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t \gg\!\gg \quad v(t) = \sqrt{(x')^2 + (y')^2} = \boxed{\sqrt{1+t}}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

Substitute $w = 1 + u$ and $dw = du$. New bounds are 1 and $1 + t$. Calculate:

$$\gg\!\gg \quad \int_1^{1+t} \sqrt{w}\, dw$$

$$\gg\!\gg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \gg\!\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\!\gg \quad \frac{2}{3}(27 - 1) \quad \gg\!\gg \quad \boxed{\frac{52}{3}}$$

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Now compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \quad \gg\!\!\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=0} \quad \gg\!\!\gg \quad (0,0)$$

For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \gg\!\!\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=8}$$

$$\gg\!\!\gg \quad \left(8, \frac{2}{3}8^{3/2}\right) \quad \gg\!\!\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Insert $(0,0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\gg\!\!\gg \quad \sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\!\!\gg \quad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\!\!\gg \quad \frac{\sqrt{2624}}{3}$$

Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

Parametrization of the unit upper semicircle:

$$(x, y) = (\cos t,\, \sin t)$$

$$\gg\!\!\gg \quad (x', y') = (-\sin t,\, \cos t)$$

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$$\gg\!\!\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \quad \gg\!\!\gg \quad dt$$

Now for $R(t)$ we choose $R(t) = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t\, dt \quad \gg\!\!\gg \quad -2\pi \cos t\Big|_0^\pi \quad \gg\!\!\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3,\, t^2 - 1)$ for $0 \le t \le 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \ggg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \ggg \quad \sqrt{9t^4 + 4t^2}$$

$$\ggg \quad \sqrt{t^2(9t^2 + 4)} \quad \ggg \quad t\sqrt{9t^2 + 4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \ggg \quad \int_0^1 2\pi(t^2 - 1)t\sqrt{9t^2 + 4}\, dt$$

Converting to polar pi-correction

Compute the polar coordinates of $\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \ggg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \ggg \quad -\pi/3$$

This angle is in Quadrant IV. We *add π* to get the polar angle in Quadrant II:

$$\theta = \pi - \pi/3 \quad \ggg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r, \theta) = (1, 2\pi/3)$.

For $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV. (No extra π needed.)

Next compute:

$$\tan^{-1}\left(\frac{-\sqrt{2}/2}{+\sqrt{2}/2}\right) \quad \ggg \quad \tan^{-1}(-1) \quad \ggg \quad -\pi/4$$

So the point in polar is $(1, -\pi/4)$.

Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y - 3)^2 = 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \qquad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \quad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

$$\gg\gg \qquad r^2 - 6r\sin\theta = 0 \qquad \gg\gg \qquad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.