W14 - Notes

More calculus with parametric curves

07 Theory - Distance, speed

Distance function

The **distance function** $s(t)$ returns the total distance traveled by the particle from a chosen starting time t_0 up to the (input) time t:

$$s(t) \;=\; \int_{t_0}^{t} ds \quad = \quad \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

We need the dummy variable u so that the integration process does not conflict with t in the upper bound.

Speed function

The **speed** of a moving particle is the *rate of change of distance*:

$$v(t) \;=\; s'(t) \quad = \quad \sqrt{x'(t)^2 + y'(t)^2}$$

This formula can be explained in either of two ways:

1. Apply the Fundamental Theorem of Calculus to the integral formula for $s(t)$.
2. Consider $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ for a small change dt: so the *rate of change* of arclength is $\frac{ds}{dt}$, in other words $s'(t)$.

08 Illustration

Example - Speed, distance, displacement

The parametric curve $\left(t, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)

Compute *derivatives*:

$$(x',\, y') = \left(1,\, t^{1/2}\right)$$

Now compute the *speed*:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t \ggg \quad v(t) = \sqrt{(x')^2 + (y')^2} = \boxed{\sqrt{1+t}}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

Substitute $w = 1 + u$ and $dw = du$. New bounds are 1 and $1 + t$. Calculate:

$$\ggg \quad \int_{1}^{1+t} \sqrt{w}\, dw$$

$$\ggg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \ggg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \ggg \quad \frac{2}{3}(27 - 1) \quad \ggg \quad \boxed{\frac{52}{3}}$$

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Now compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \quad \ggg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=0} \quad \ggg \quad (0, 0)$$

For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \ggg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=8}$$

$$\ggg \quad \left(8, \frac{2}{3}8^{3/2}\right) \quad \ggg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Insert $(0, 0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\gg\gg \quad \sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\gg \quad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\gg \quad \boxed{\frac{\sqrt{2624}}{3}}$$

09 Theory - Surface area of revolutions

Surface area of a surface of revolution: thin bands

Suppose a parametric curve $\big(x(t),\, y(t)\big)$ is revolved around the x-axis or the y-axis.

The surface area is:

$$A \;=\; \int_a^b 2\pi R(t)\,\sqrt{(x')^2 + (y')^2}\,dt$$

The radius $R(t)$ should be the distance to the axis:

$$\begin{aligned} R(t) \;&=\; y(t) \qquad \text{revolution about } x\text{-axis} \\ R(t) \;&=\; x(t) \qquad \text{revolution about } y\text{-axis} \end{aligned}$$

This formulas adds the areas of thin bands, but the bands are demarcated using parametric functions instead of input values of a graphed function.

The formula assumes that the curve is traversed one time as t increases from a to b.

10 Illustration

Example - Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

Parametrization of the unit upper semicircle:

$$(x, y) = (\cos t,\, \sin t)$$

$$\gg\gg \quad (x', y') = (-\sin t,\, \cos t)$$

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\,dt$$

$$\gg\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt \quad \gg\gg \quad dt$$

Now for $R(t)$ we choose $R(t) = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t \, dt \quad \gg\gg \quad -2\pi \cos t \Big|_0^\pi \quad \gg\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1-x^2}$.

Example - Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve (t^3, t^2-1) for $0 \leq t \leq 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \gg\gg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \gg\gg \quad \sqrt{9t^4 + 4t^2}$$

$$\gg\gg \quad \sqrt{t^2(9t^2+4)} \quad \gg\gg \quad t\sqrt{9t^2+4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R \, ds \quad \gg\gg \quad \int_0^1 2\pi(t^2-1)t\sqrt{9t^2+4}\, dt$$

Polar curves

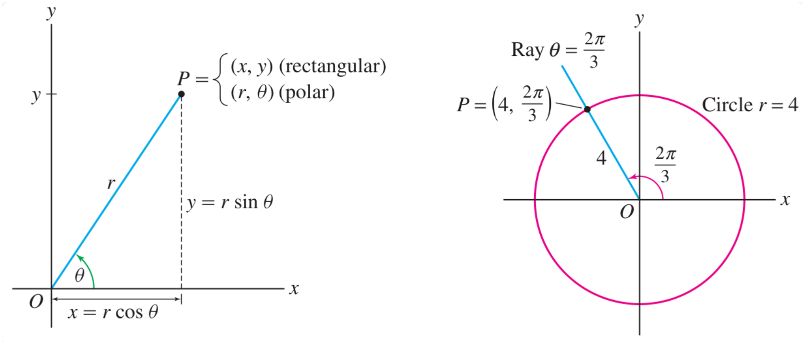
Videos

Videos, Organic Chemistry Tutor

- Polar coordinates intro
- Graphing polar curves

01 Theory - Polar points, polar curves

Polar coordinates are pairs of numbers (r, θ) which identify points in the plane in terms of *distance to origin* and *angle from $+x$-axis*:



Converting Polar ↔ Cartesian

Polar → Cartesian

$x = r\cos\theta$

$y = r\sin\theta$

Cartesian → Polar

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \dfrac{y}{x} \quad (x \neq 0)$

Polar coordinates have *many redundancies*: unlike Cartesian which are unique!

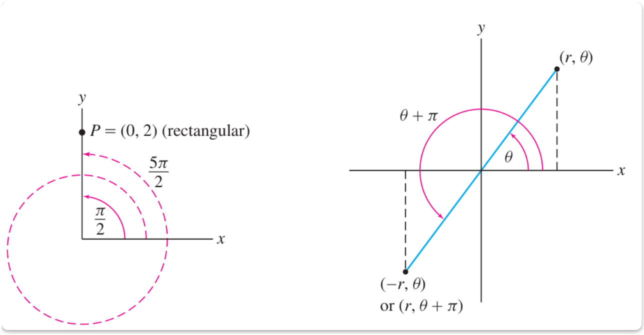
- For example: $(r, \theta) = (r, \theta + 2\pi)$
  - And therefore also $(r, \theta) = (r, \theta - 2\pi)$ (negative θ can happen)
- For example: $(-r, \theta) = (r, \theta + \pi)$ for every r, θ
- For example: $(0, \theta) = (0, 0)$ for any θ

Polar coordinates *cannot be added*: they are not vector components!

- For example $(5, \pi/3) + (2, \pi/6) \neq (7, \pi/2)$
- Whereas Cartesian coordinates can be added: $(1, 4) + (2, -2) = (3, 2)$

⚠ **The transition formulas** Cartesian → Polar **require careful choice of θ.**

- The standard definition of $\tan^{-1}\left(\frac{y}{x}\right)$ sometimes gives *wrong θ*
  - This is because it uses the restricted domain $\theta \in (-\pi/2, \pi/2)$; the polar interpretation is: only points in Quadrant I and Quadrant IV (SAFE QUADRANTS)
- Therefore: *check signs* of x and y to see *which quadrant*, maybe need π-correction!
  - Quadrant I or IV: polar angle is $\tan^{-1}\left(\frac{y}{x}\right)$
  - Quadrant II or III: polar angle is $\tan^{-1}\left(\frac{y}{x}\right) + \pi$



Equations (as well as points) can also be converted to polar.

For Cartesian → Polar, look for cancellation from $\cos^2\theta + \sin^2\theta = 1$.

For Polar → Cartesian, try to keep θ inside of trig functions.

- For example:

$$r = \sin^2\theta \qquad \gg\gg \qquad \sqrt{x^2+y^2} = \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2$$

02 Illustration

Example - Converting to polar: π-correction

Compute the polar coordinates of $\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \gg\gg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \gg\gg \quad -\pi/3$$

This angle is in Quadrant IV. We *add π* to get the polar angle in Quadrant II:

$$\theta = \pi - \pi/3 \quad \gg\gg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r,\theta) = (1, 2\pi/3)$.

For $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV. (No extra π needed.)

Next compute:

$$\tan^{-1}\left(\frac{-\sqrt{2}/2}{+\sqrt{2}/2}\right) \quad \gg\gg \quad \tan^{-1}(-1) \quad \gg\gg \quad -\pi/4$$

So the point in polar is $(1, -\pi/4)$.

Example - Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y-3)^2 = 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \qquad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \quad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

$$\gg\gg \qquad r^2 - 6r\sin\theta = 0 \qquad \gg\gg \qquad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.

03 Theory - Polar limaçons

To draw the polar graph of some function, it can help to first draw the Cartesian graph of the function. (In other words, set $y = r$ and $x = \theta$, and draw the usual graph.) By tracing through the points on the Cartesian graph, one can visualize the trajectory of the polar graph.

This Cartesian graph may be called a **graphing tool** for the polar graph.

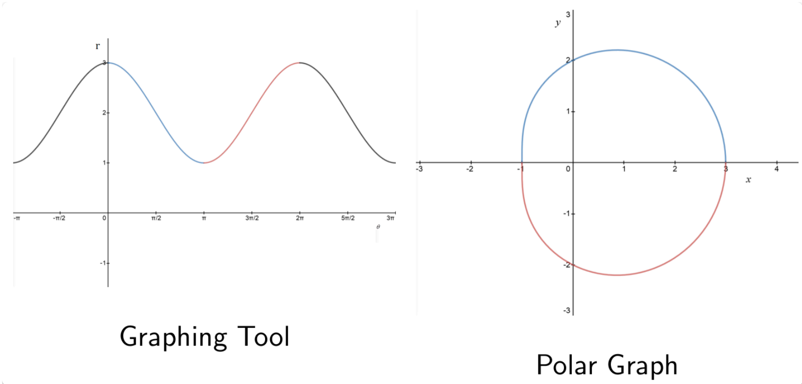
A limaçon is the polar graph of $r(\theta) \;=\; a + b\cos\theta$.

The *shape* of a limaçon is determined by the value of $c = \frac{b}{a}$. Any limaçon can be rescaled to have this form:

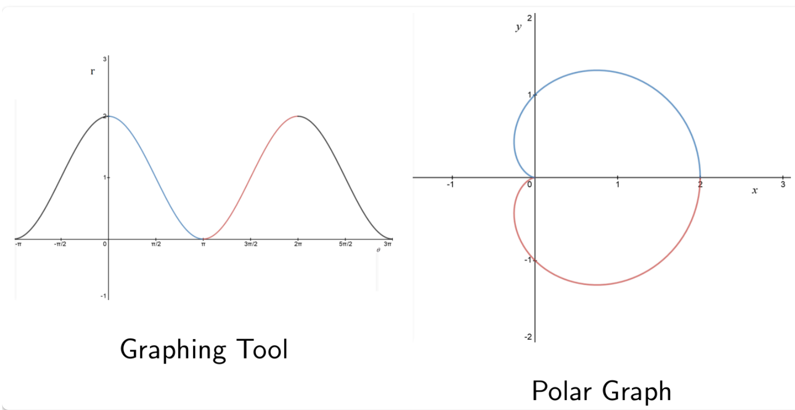
$$r = 1 + c\cos\theta$$

Limaçon satisfying $r(\theta) = 1$: unit circle.

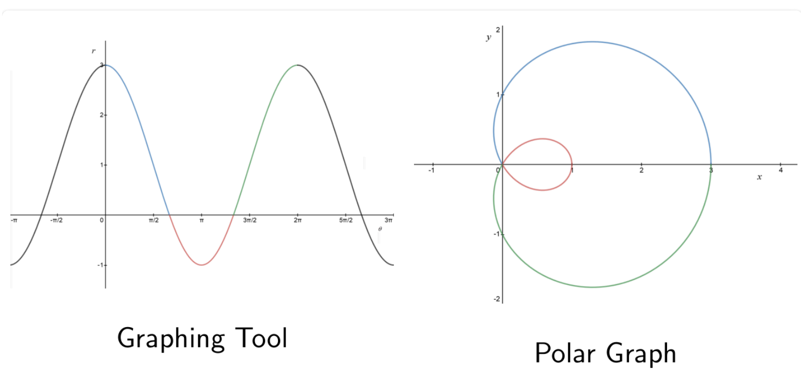
Limaçon satisfying $r(\theta) = 2 + \cos\theta$: 'outer loop' circle with 'dimple':



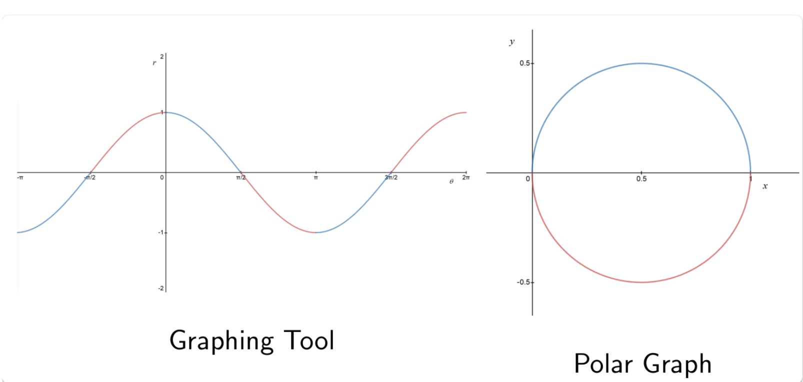
Limaçon satisfying $r(\theta) = 1 + \cos\theta$: 'cardioid' = 'outer loop' circle with 'dimple' that creates a cusp:



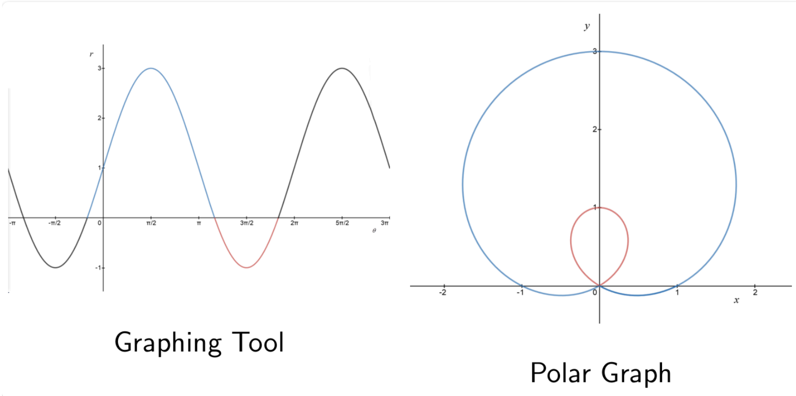
Limaçon satisfying $r(\theta) = 1 + 2\cos\theta$: 'dimple' pushes past cusp to create 'inner loop':



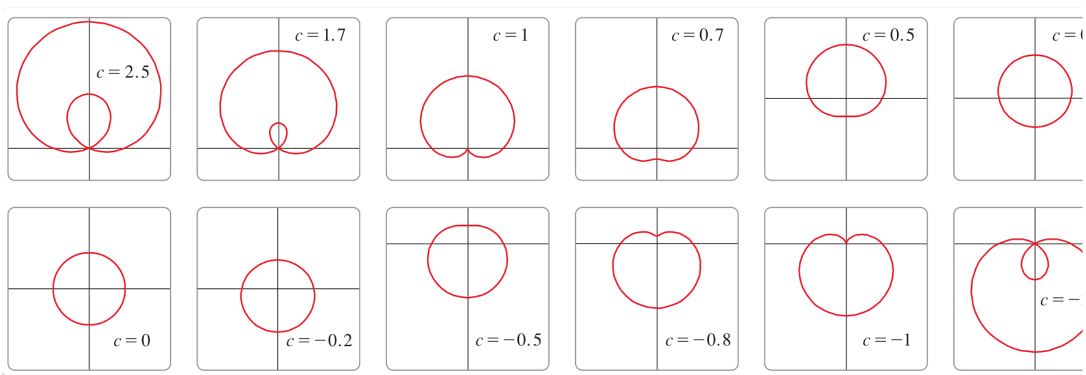
Limaçon satisfying $r(\theta) = \cos\theta$: 'inner loop' only, no outer loop exists:



Limaçon satisfying $r(\theta) = 1 + 2\sin\theta$: 'inner loop' and 'outer loop' and rotated ⟲ $90°$:



Transitions between limaçon types, $r(\theta) \;=\; 1 + c\sin\theta$:



Notice the transition points at $|c| = 0.5$ and $|c| = 1$:

The *flat spot* occurs when $c = \pm 0.5$

- Smaller c gives *convex shape*

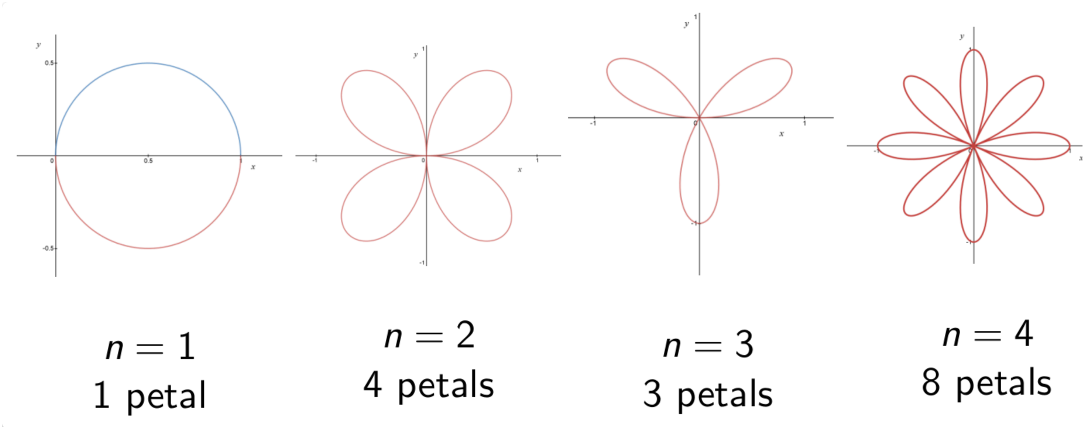
The *cusp* occurs when $c = \pm 1$

- Smaller c gives *dimple* (assuming $|c| > 0.5$)
- Larger c gives *inner loop*

04 Theory - Polar roses

Roses are polar graphs of this form:

$$r \;=\; \cos(\theta), \qquad r \;=\; \sin(2\theta), \qquad r \;=\; \sin(3\theta), \qquad r \;=\; \cos(4\theta)$$



The pattern of petals:

- $n = 2k$ (even): obtain $2n$ petals
  - These petals traversed *once*
- $n = 2k + 1$ (odd): obtain n petals
  - These petals traversed *twice*
- Either way: total-petal-traversals: always $2n$

05 Illustration

Example - Finding vertical tangents to a limaçon

Let us find the vertical tangents to the limaçon (the cardioid) given by $r = 1 + \sin\theta$.

Solution

(1) Convert to Cartesian parametric using $x = r\cos\theta$ and $y = r\sin\theta$:

$$r(\theta) = 1 + \sin\theta \quad \gg\gg \quad (x, y) = \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)$$

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(2) Compute x' and y':

$$(x'(\theta),\, y'(\theta))$$

$$\gg\gg \quad \Big(\cos\theta\cos\theta + (1 + \sin\theta)(-\sin\theta),\ \cos\theta\sin\theta + (1 + \sin\theta)\cos\theta\Big)$$

$$\gg\gg \quad \Big(\cos^2\theta - \sin^2\theta - \sin\theta,\ \cos\theta\,(1 + 2\sin\theta)\Big)$$

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(3) The vertical tangents occur when $x'(\theta) = 0$. We must double check that $y'(\theta) \neq 0$ at these points.

$$x'(\theta) = 0 \quad \gg\gg \quad \cos^2\theta - \sin^2\theta - \sin\theta = 0$$

$$\gg\gg \quad (1 - \sin^2\theta) - \sin^2\theta - \sin\theta = 0$$

Substitute $A = \sin\theta$ and observe quadratic:

$$\gg\gg \quad 1 - 2A^2 - A = 0 \quad \gg\gg \quad 2A^2 + A - 1 = 0$$

Solve:

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \gg\gg$$

$$\frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} \quad \gg\gg \quad \frac{1}{2}, -1$$

Then find θ:

$$A = \sin\theta \quad \gg\gg \quad \sin\theta = \frac{1}{2}, -1$$

$$\gg\gg \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \text{ (for } 1/2) \quad \text{and} \quad \theta = \frac{3\pi}{2} \text{ (for } -1)$$

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(4) Compute the points. In polar coordinates:

$$(r, \theta) = (1 + \sin\theta, \theta)\Big|_{\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}}$$

$$\gg\gg \quad \left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), \left(0, \frac{3\pi}{2}\right)$$

In Cartesian coordinates:

At $\theta = \frac{\pi}{6}$:

$$(x, y)\Big|_{\theta = \frac{\pi}{6}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta = \frac{\pi}{6}}$$

$$\gg\gg \quad \left(\left(1 + \frac{1}{2}\right)\frac{\sqrt{3}}{2},\ \left(1 + \frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$$

At $\theta = \frac{5\pi}{6}$:

$$(x, y)\Big|_{\theta = \frac{5\pi}{6}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta = \frac{5\pi}{6}}$$

$$\gg\gg \quad \left(\left(1 + \frac{1}{2}\right)\frac{-\sqrt{3}}{2},\ \left(1 + \frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(-\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$$

At $\theta = \frac{3\pi}{2}$:

$$(x, y)\Big|_{\theta = \frac{3\pi}{2}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta = \frac{3\pi}{2}}$$

$$\gg\gg \quad \Big((1 - 1) \cdot 0,\ (1 - 1) \cdot (-1)\Big) \quad \gg\gg \quad (0, 0)$$

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(5) Correction: $(0,0)$ is a cusp!

The point $(0,0)$ at $\theta = \frac{3\pi}{2}$ is on the cardioid, but the curve is not smooth there, this is a cusp.

Still, the left- and right-sided tangents exists and are equal, so in a certain sense we could say the curve has vertical tangent at $\theta = \frac{3\pi}{2}$.