Limaçons & Roses

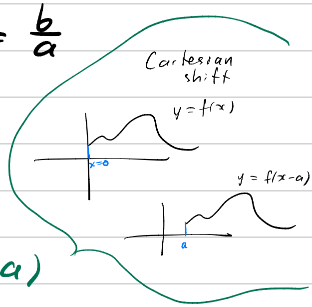
Look at graphs of $r(\theta) = a + b\cos\theta$, $a + b\sin\theta$
"Limaçons"

and $r(\theta) = \cos(n\theta)$, $\sin(n\theta)$
"rose"

Limaçon $r = a + b\cos\theta$, shape $c = \frac{b}{a}$

$\div$ by a

can rescale to $r = 1 + \frac{b}{a}\cos\theta$



Note: $r = f(\theta)$	$r = f(\theta)$
$\rightsquigarrow$ $r = a f(\theta)$	$\rightsquigarrow$ $r = f(\theta - a)$
$a > 1$: expand	rotate by a
$a < 1$: shrink	counterclockwise

[See pictures in web notes.]

4/14

Calculus with Polar Curves

Some $r = r(\theta)$, $\alpha \le \theta \le \beta$

vs. Cartesian: $L = \int_a^b \sqrt{1+f'^2}\,dx$ $\quad \sqrt{dx^2+dy^2}$ $\quad y = f(x)$

Polar arclength: $$L = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\,d\theta$$

i.e. $\sqrt{r'^2 + r^2}\,d\theta$

Why? Convert to parametric: $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$

Then $$L = \int_\alpha^\beta \sqrt{x'^2 + y'^2}\,dt \qquad (\text{use } \theta \text{ for } t)$$

$$\rightsquigarrow \int_\alpha^\beta \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\,d\theta$$

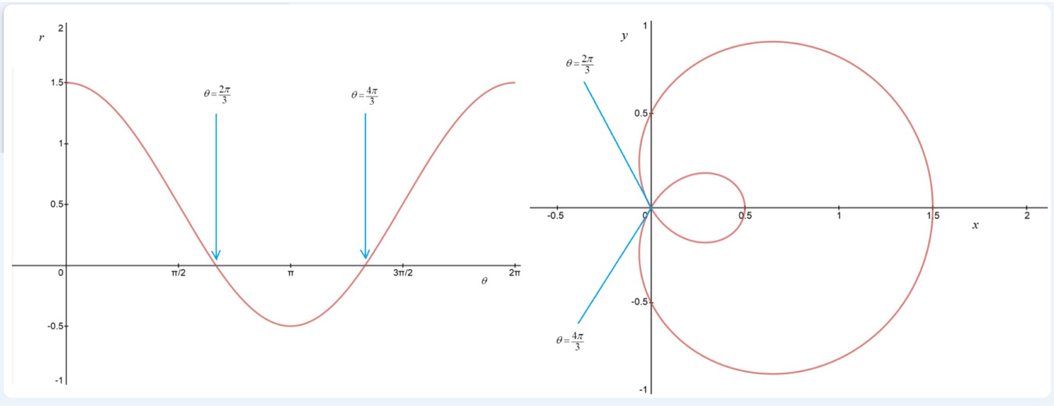
$$\rightsquigarrow \int_\alpha^\beta \sqrt{r'^2\cos^2\theta + r^2\sin^2\theta \cancel{- 2rr'\cos\theta\sin\theta} + r'^2\sin^2\theta + r^2\cos^2\theta \cancel{+ 2rr'\sin\theta\cos\theta}}\,d\theta$$

$$\rightsquigarrow \int_\alpha^\beta \sqrt{r'^2(\cos^2 + \sin^2) + r^2(\sin^2 + \cos^2)}\,d\theta$$

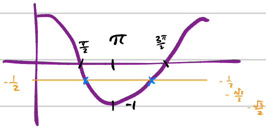
$$\rightsquigarrow \boxed{\int_\alpha^\beta \sqrt{r'^2 + r^2}\,d\theta}$$

Example: Limaçon $r = \frac{1}{2} + \cos\theta$.
How long is the inner loop? Set up integral.

Solution:



$\frac{1}{2} + \cos\theta = 0 \rightsquigarrow \cos\theta = -\frac{1}{2}, \quad \theta =$



$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

$\rightsquigarrow \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

$\rightsquigarrow \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

$$L = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{(-\sin\theta)^2 + \left(\tfrac{1}{2} + \cos\theta\right)^2}\, d\theta$$

$$\rightsquigarrow \int_{\frac{2\pi}{3}}^{4\pi/3} \sqrt{\sin^2\theta + \cos^2\theta + \tfrac{1}{4} + 2\cdot\tfrac{1}{2}\cdot\cos\theta}\ d\theta$$

$$\rightsquigarrow \boxed{\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{5/4 + \cos\theta}\ d\theta}$$

Polar Area

$r = r(\theta)$ or $r_0(\theta),\ r_1(\theta)$ $\qquad \alpha \le \theta \le \beta$

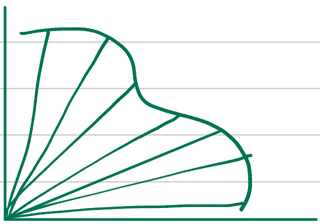
$\int_a^b f\,dx$

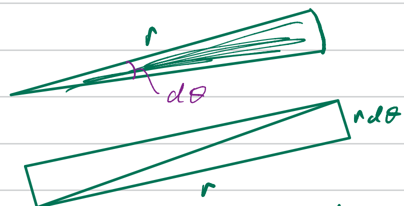
$$A = \int_\alpha^\beta \frac{1}{2} r(\theta)^2 \, d\theta$$ sector "below" $r(\theta)$

$\int_a^b f-g\,dx$

$$A = \int_\alpha^\beta \tfrac{1}{2}(r_1^2 - r_0^2)\, d\theta$$ sector "between" $r_0(\theta),\ r_1(\theta)$

Why?





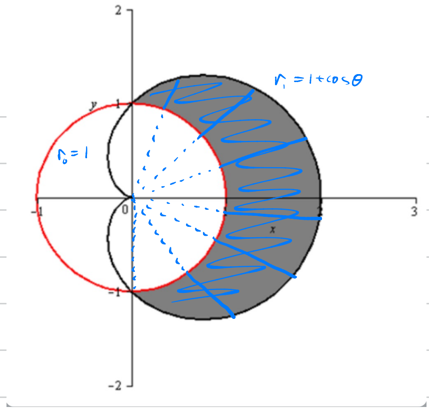
$\sim \frac{1}{2} r^2 d\theta$

Alternate: portion of circle:

$$\underset{\text{circle}}{(\pi r^2)}\ \underset{\%}{\left(\frac{d\theta}{2\pi}\right)} = \frac{1}{2} r^2 d\theta$$

Example: Find area shaded:

Limaçon: $r = 1+\cos\theta$



$$-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_1^2 - r_0^2\right)\, d\theta$$

$$\rightsquigarrow \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{2}\left((1+\cos\theta)^2 - (1)^2\right) d\theta$$

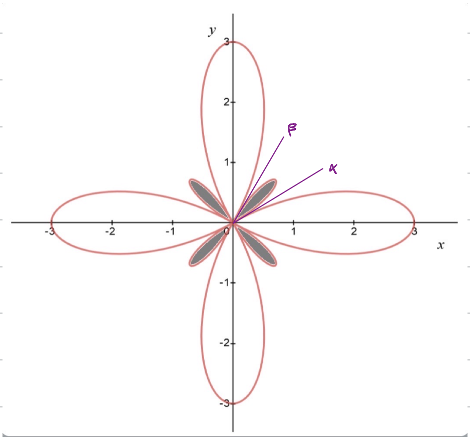
$$\rightsquigarrow \frac{1}{2}\int_{-\pi/2}^{+\pi/2} 1 + \cos^2\theta + 2\cos\theta - 1 \, d\theta \quad \rightsquigarrow \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2\cos\theta + \cos^2\theta \, d\theta$$

$$\rightsquigarrow \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2\cos\theta + \frac{1}{2}\left(1+\cos(2\theta)\right) d\theta$$

$$\rightsquigarrow -2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \quad \rightsquigarrow \boxed{2+\frac{\pi}{4}}$$

Example: $r = 1 + 2\cos(4\theta)$

Find are of inner loop.



Solution: $1 + 2\cos(4\theta) = 0$

$\rightsquigarrow \cos(4\theta) = -\frac{1}{2}$

$\rightsquigarrow 4\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (some effort required...)

$\rightsquigarrow \theta = \underset{\alpha}{\frac{\pi}{6}}, \underset{\beta}{\frac{\pi}{3}}$

4x small loops

$$A = 4 \cdot \int_{\pi/6}^{\pi/3} \frac{1}{2} \underbrace{(1+2\cos(4\theta))^2}_{} d\theta$$

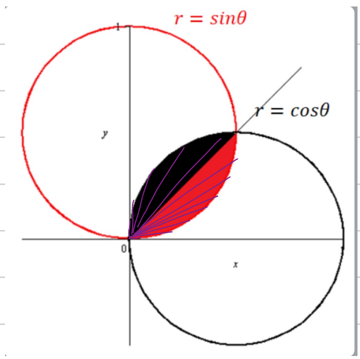
$\to \frac{1}{2}(1 + 4\cos(4\theta) + 4\cos^2(4\theta))$

$\rightsquigarrow \boxed{\pi - \frac{3\sqrt{3}}{2}}$

$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$

$\cos^2(4\theta) = \frac{1}{2}(1+\cos(8\theta))$

Example: $r_1 = \sin\theta, \quad r_2 = \cos\theta$



$$A = 2\int_0^{\pi/4} \frac{1}{2}(\sin\theta)^2 d\theta = 2(\text{red area})$$

$$\rightsquigarrow \int_0^{\pi/4} \sin^2\theta \, d\theta$$

$$\rightsquigarrow \int_0^{\pi/4} \frac{1}{2}(1-\cos(2\theta))\,d\theta \rightsquigarrow \frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/4} \rightsquigarrow \boxed{\frac{\pi}{8} - \frac{1}{4}}$$