W15 - Examples

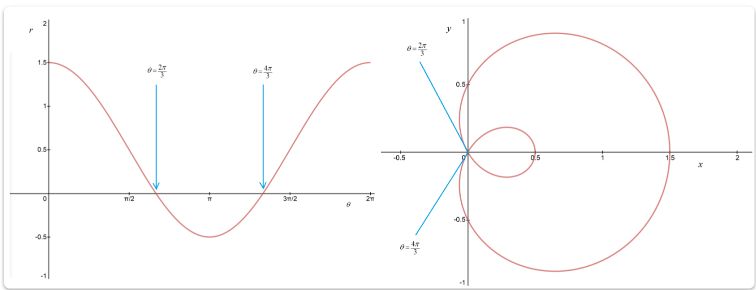
Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos\theta$.

How long is the inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

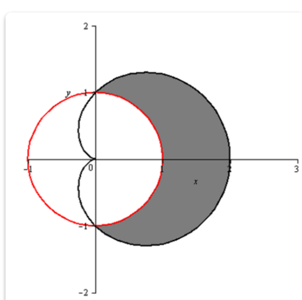
$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(-\sin\theta)^2 + \left(\frac{1}{2} + \cos\theta\right)^2}\, d\theta \quad \gg\gg \quad \int_{2\pi/3}^{4\pi/3} \sqrt{5/4 + \cos\theta}\, d\theta$$

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(\theta) = 1$ and the limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution

First draw the region:



The two curves intersect at $\theta = \pm\frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}$:

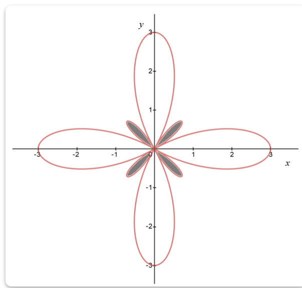
$$A = \int_a^b \frac{1}{2}(r_1^2 - r_0^2)\,d\theta \quad \gg\!\gg \quad \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - 1^2\Big)\,d\theta$$

$$\gg\!\gg \quad \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta\,d\theta \quad \gg\!\gg \quad \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{4}\big(1+\cos(2\theta)\big)\,d\theta$$

$$\gg\!\gg \quad \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \quad \gg\!\gg \quad \boxed{2 + \frac{\pi}{4}}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

Find bounds for one small loop. Lower left loop occurs first. This loop is when $1 + 2\cos(4\theta) \le 0$.

$$1 + 2\cos(4\theta) \le 0 \quad \gg\!\gg \quad \cos(4\theta) \le -\frac{1}{2}$$

$$\gg\!\gg \quad \frac{2\pi}{3} \le 4\theta \le \frac{4\pi}{3} \quad \gg\!\gg \quad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

Now set up area integral:

$$A = 4\int_\alpha^\beta \frac{1}{2} r(\theta)^2\,d\theta \quad \gg\!\gg \quad 4\int_{\pi/6}^{\pi/3} \frac{1}{2}\big(1 + 2\cos(4\theta)\big)^2\,d\theta$$

$$\gg\!\gg \quad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4\cos^2(4\theta)\,d\theta$$

Power-to-frequency conversion: $\cos^2 A \rightsquigarrow \frac{1}{2}(1 + \cos(2A))$ with $A = 4\theta$:

$$\gg\!\gg \quad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4\cdot\frac{1}{2}\big(1 + \cos(8\theta)\big)\,d\theta$$

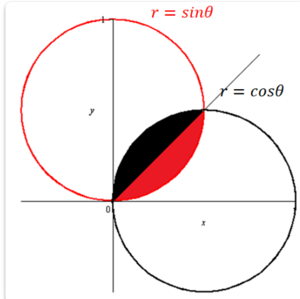
$$\gg\!\gg \quad 6\theta + 2\sin(4\theta) + \frac{1}{4}\sin(8\theta)\Big|_{\pi/6}^{\pi/3} \quad \gg\!\gg \quad \boxed{\pi - \frac{3\sqrt{3}}{2}}$$

Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta) = \sin\theta$ and the other is given by $r(\theta) = \cos\theta$.

Solution

Drawing of the overlap:



Notice: total overlap area = 2× area of red region. Bounds for red region: $0 \leq \theta \leq \frac{\pi}{4}$.

Area formula applied to $r(\theta) = \sin\theta$:

$$A \; = \; 2\int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta \quad \gg\!\gg \quad 2\int_0^{\pi/4} \frac{1}{2}\sin^2\theta \, d\theta$$

Power-to-frequency: $\sin^2\theta \rightsquigarrow \frac{1}{2}\big(1 - \cos(2\theta)\big)$:

$$\gg\!\gg \quad 2\int_0^{\pi/4} \frac{1}{4}\big(1 - \cos(2\theta)\big) \, d\theta$$

$$\gg\!\gg \quad \frac{2}{4}\theta - \frac{2}{8}\sin(2\theta)\Big|_0^{\pi/4} \quad \gg\!\gg \quad \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Complex multiplication

Compute the products:

(a) $(1 - i)(4 - 7i)$ (b) $(2 + 5i)(2 - 5i)$

Solution

(a) $(1 - i)(4 - 7i)$

Expand:

$$(1 - i)(4 - 7i) \quad \gg\!\gg \quad 4 - 7i - 4i + 7i^2$$

Simplify i^2:

$$\gg\!\gg \quad 4 - 7i - 4i + 7(-1) \quad \gg\!\gg \quad \boxed{-3 - 11i}$$

(b) $(2 + 5i)(2 - 5i)$

Expand:

$$(2 + 5i)(2 - 5i) \quad \gg\!\gg \quad 4 - 10i + 10i - 25i^2$$

Simplify i^2:

$$\gg\!\gg \quad 4 - 10i + 10i - 25(-1) \quad \gg\!\gg \quad \boxed{29}$$

Complex division

Compute the following divisions of complex numbers:

(a) $\frac{1}{-3+i}$ (b) $\frac{1}{i}$ (c) $\frac{1}{7i}$ (d) $\frac{2+5i}{2-5i}$

Solution

(a) $\frac{1}{-3+i}$

Conjugate is $-3-i$:

$$\frac{1}{-3+i} \quad \ggg \quad \frac{1}{-3+i} \cdot \frac{-3-i}{-3-i}$$

Simplify:

$$\ggg \quad \frac{-3-i}{9+1} \quad \ggg \quad \boxed{\frac{-3}{10} + \frac{-1}{10} i}$$

(b) $\frac{1}{i}$

Conjugate is $-i$:

$$\frac{1}{i} \quad \ggg \quad \frac{1}{i} \cdot \frac{-i}{-i} \quad \ggg \quad -i$$

(c) $\frac{1}{7i}$

Factor out the $1/7$:

$$\frac{1}{7i} \quad \ggg \quad \boxed{\frac{1}{7} \cdot \frac{1}{i}}$$

Use $\frac{1}{i} = -i$:

$$\ggg \quad \frac{1}{7} \cdot (-i) \quad \ggg \quad \frac{-1}{7} i$$

(d) $\frac{2+5i}{2-5i}$

Denominator conjugate is $2+5i$:

$$\frac{2+5i}{2-5i} \quad \ggg \quad \frac{2+5i}{2-5i} \cdot \frac{2+5i}{2+5i}$$

Simplify:

$$\ggg \quad \frac{4+20i+25i^2}{4+25} \quad \ggg \quad \boxed{\frac{-21}{29} + \frac{20}{29} i}$$