Complex Numbers

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$z = a + bi \qquad a, b \in \mathbb{R} \qquad z \in \mathbb{C}$

$Re(z) = a, \quad Im(z) = b$

To add, subtract, multiply, divide:
- treat "i" like "x", collect like terms
- use "$i^2 = -1$" to simplify

Example: $(1+3i)(2-2i) \xrightarrow{\text{FOIL}} 2 - 2i + 6i - 6i^2$
$\xrightarrow{i^2=-1} \boxed{8 + 4i}$

Division: $\dfrac{1}{-3+i} = \left(\dfrac{1}{-3+i}\right)\left(\dfrac{-3-i}{-3-i}\right)$ "complex conjugate" of $-3+i$

$\rightsquigarrow \dfrac{-3-i}{(-3+i)(-3-i)} \rightsquigarrow \dfrac{-3-i}{9+\cancel{3i}-\cancel{3i}-i^2}$

$\rightsquigarrow \dfrac{-3-i}{9-(-1)} \rightsquigarrow \boxed{\left(\dfrac{-3}{10}\right) + \left(\dfrac{-1}{10}\right) i}$

Complex Conjugates:

$z = a + bi \iff \bar{z} = a - bi$

e.g. $2+5i \iff \overline{2+5i} = 2-5i$
$2-5i \iff \overline{2-5i} = 2+5i$

$$z \cdot \bar{z} = (a+bi)(a-bi)$$

$$\rightsquigarrow a^2 - abi + bai - b^2 i^2$$

$$\rightsquigarrow a^2 + b^2 \geq 0 \qquad (=0 \Rightarrow a=0 \text{ \& } b=0)$$

So $\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{a-bi}{a^2+b^2} = \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right) i$

Examples: (a) $(1-i)(4-7i)$ (b) $(2+5i)(2-5i)$

(a) $= 4 - 7i - 4i + 7i^2 \rightsquigarrow -3 - 11i$

(b) $= 2^2 + 5^2 \rightsquigarrow 4 + 25 \rightsquigarrow 29$

Examples: (a) $\frac{1}{-3+i}$ (b) $\frac{1}{i}$ (c) $\frac{1}{7i}$ (d) $\frac{2+5i}{2-5i}$

(a) $= \frac{-3-i}{9+1} \rightsquigarrow -\frac{3}{10} - \frac{1}{10} i$

(b) $\frac{-i}{1} = -i$

$i = 0 + 1 \cdot i$

$\bar{i} = 0 - 1 \cdot i = -i$

(c) $= \frac{1}{7} \cdot \frac{1}{i} = -\frac{1}{7} i$

(d) $\frac{(2+5i)(2+5i)}{4+25} = \frac{4 + 10i + 10i - 25}{29}$

$\rightsquigarrow -\frac{21}{29} + \frac{20}{29} i$

Quadratic Formula

Solve: $z^2 + 2z + 2 = 0 \rightsquigarrow z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$

$\rightsquigarrow -1 \pm \frac{1}{2}\sqrt{-4} = -1 \pm \frac{1}{2}\sqrt{4}\sqrt{-1} = -1 \pm i$

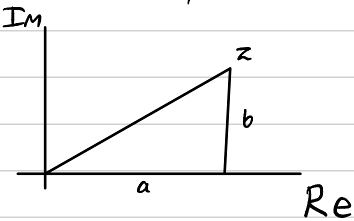
$$\boxed{z = -1+i, \; -1-i}$$

Complex Exponential

Complex Plane:

Cartesian:

$a+bi$ and (a,b)



$z = a+bi$

$$r = \sqrt{a^2+b^2} \geq 0$$
$$= |z|$$
$$= \text{"modulus" of } z$$

$$\theta = \tan^{-1}(b/a) \overset{?}{+} \pi$$
$$= \operatorname{Arg}(z)$$
$$= \text{"argument" of } z$$

$$z\bar{z} = |z|^2 = a^2+b^2$$

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cis form:

Recall $a = r\cos\theta$, $b = r\sin\theta$

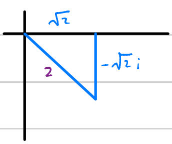
So $z = a+bi = r\cos\theta + r\sin\theta\, i$

$$\rightsquigarrow r(\cos\theta + i\sin\theta)$$
$$\rightsquigarrow r\operatorname{cis}\theta$$

(Same data as polar data (r,θ).)

Example:

$\sqrt{2} - \sqrt{2}i = 2\operatorname{cis}(-\frac{\pi}{4})$ from figure



also: $\hookrightarrow = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$$\rightsquigarrow 2\left(\cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4})\right)$$
$$\rightsquigarrow 2\operatorname{cis}(-\tfrac{\pi}{4})$$

Complex Exponential

Euler Formula: $re^{i\theta} = r\,\mathrm{cis}\,\theta$

$$= r\cos\theta + r\sin\theta\, i$$

when $r=1 \rightsquigarrow e^{i\theta} = \mathrm{cis}\,\theta$

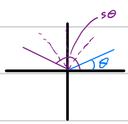
when also $\theta = \pi \rightsquigarrow \boxed{e^{i\pi} = -1}$

$$e^{i\pi} + 1 = 0$$

Products & Powers

$z = re^{i\theta}$, $\quad z^n = (re^{i\theta})^n$

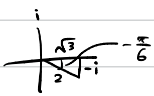
$$\rightsquigarrow r^n e^{in\theta}$$



Interpret geometrically:

power n
- stretch $r \mapsto r^n$
- rotate $\theta \mapsto n\theta$

Example: (a) $(\sqrt{3} - i)^{72}$



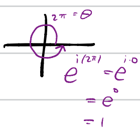
$\rightsquigarrow \sqrt{3} - i = 2e^{i(-\frac{\pi}{6})}$

$\rightsquigarrow (\sqrt{3} - i) = \left(2e^{i(-\frac{\pi}{6})}\right)^{72} \rightsquigarrow 2^{72} e^{-12\pi i}$

$\rightsquigarrow 2^{72} e^{(2\pi i)\cdot(-6)}$

$\rightsquigarrow 2^{72} (e^{2\pi i})^{-6}$

$\rightsquigarrow 2^{72} \cdot 1^{-6} \rightsquigarrow \boxed{2^{72}}$



(b) $(\sqrt{3}+i)^{66} = ?$

$$\sqrt{3}+i = 2e^{i\frac{\pi}{6}} \rightsquigarrow (\sqrt{3}+i)^{66} = \left(2e^{i\frac{\pi}{6}}\right)^{66}$$

$$\rightsquigarrow 2^{66} e^{11\pi i} \rightsquigarrow 2^{66} e^{(2\pi\cdot 5+\pi)i}$$

$$\rightsquigarrow 2^{66}\left(e^{2\pi i}\right)^{5} e^{\pi i}$$

$$\rightsquigarrow 2^{66}\cdot 1\cdot(-1) \rightsquigarrow \boxed{-2^{66}}$$

$$(\sqrt{3}+i)^{69} \rightsquigarrow 2^{69} e^{i\frac{69\pi}{6}}$$

$$\rightsquigarrow 2^{69} e^{i\left(10\pi+\frac{3\pi}{2}\right)}$$

$$\rightsquigarrow \boxed{-2^{69} i}$$

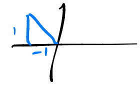
$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1\cdot z_2 = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$$

$$z_1/z_2 = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$$

Multiply/Divide <u>modulus</u>

Add/Subtract <u>argument</u>



Example: $(\sqrt{3}+i)(-1+i)$

$$2e^{i\frac{\pi}{6}} \times \sqrt{2}\,e^{i\,3\pi/4}$$

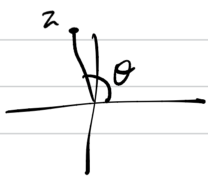
$$= \boxed{2\sqrt{2}\,e^{i\frac{11\pi}{12}}}$$

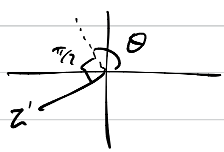
Example: Q: find $w = c+di$ such that:

$w \cdot z$ means: rotate z 90° ccw.

A: $z = re^{i\theta}$

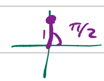
rotated 90° ccw $= re^{i(\theta+\frac{\pi}{2})} = z'$





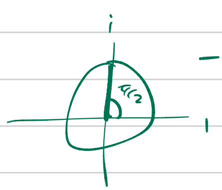
$$w \cdot re^{i\theta} = re^{i(\theta+\frac{\pi}{2})} \qquad \text{find } w.$$

$$\rightsquigarrow \quad w = \frac{re^{i(\theta+\frac{\pi}{2})}}{re^{i\theta}} \rightsquigarrow e^{i(\theta+\frac{\pi}{2})-i\theta}$$



$$\rightsquigarrow e^{i\frac{\pi}{2}} \rightsquigarrow \boxed{i}$$

$$e^{i\frac{\pi}{2}}$$



–OR–

$$\text{cis}\,\frac{\pi}{2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$
$$= 0 + i\cdot 1 = i$$

Complex Roots

Goal $\sqrt{i}$, $\sqrt{1+i}$, $\sqrt[3]{i}$, ..., $\sqrt[n]{z}$

Method: $z = re^{i\theta}$

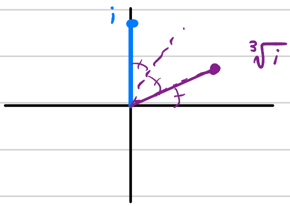
$$\sqrt[n]{z} = (re^{i\theta})^{1/n} \rightsquigarrow r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \quad \text{(gives one root)}$$

Examples:

$$\sqrt{i} = i^{\frac{1}{2}} \rightsquigarrow (e^{i\frac{\pi}{2}})^{\frac{1}{2}} \rightsquigarrow e^{i\frac{\pi}{4}} \rightsquigarrow \cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}$$
$$= \boxed{\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i}$$

$$\sqrt{1+i} = (1+i)^{\frac{1}{2}} \rightsquigarrow (\sqrt{2}e^{i\frac{\pi}{4}})^{\frac{1}{2}} \rightsquigarrow \boxed{2^{\frac{1}{4}} e^{i\frac{\pi}{8}}}$$
$$= 2^{\frac{1}{4}}\cos(\tfrac{\pi}{8}) + 2^{\frac{1}{4}}\sin(\tfrac{\pi}{8})i$$

$$\sqrt[3]{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{3}} \rightsquigarrow e^{i\frac{\pi}{6}} \rightsquigarrow \cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}$$
$$= \boxed{\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i}$$



There are actually ... n distinct n^{th} roots

(n^{th} roots = Solutions to $w^n = z$.)

Look at $z = re^{i\theta}$, n^{th} roots:

$$re^{i\theta} = re^{i(\theta+2\pi)}$$

$$\rightsquigarrow \left(re^{i\theta}\right)^{\frac{1}{n}} \overset{?}{\underset{?}{=}} \left(re^{i(\theta+2\pi)}\right)^{\frac{1}{n}}$$

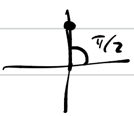
$$\rightsquigarrow r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \cancel{=} r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)} \quad ?$$

Similarly ... $\quad re^{i\theta} = re^{i(\theta+2\pi+2\pi)}$

$$r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \cancel{=} r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}+\frac{2\pi}{n}\right)}$$

Example: $\sqrt{i}$, $\sqrt[3]{i}$

$\sqrt{i}$: $\quad i = e^{i\frac{\pi}{2}} \rightsquigarrow \quad w_0 = e^{i\frac{\pi}{4}} \rightsquigarrow \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$



$w_1 = e^{i\left(\frac{\pi}{4}+\frac{2\pi}{2}\right)} \rightsquigarrow -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\sqrt[3]{i}$: $\quad i = e^{i\frac{\pi}{2}} \rightsquigarrow \quad w_0 = e^{i\frac{\pi}{6}} \rightsquigarrow \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_1 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} \rightsquigarrow -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_2 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}+\frac{2\pi}{3}\right)} \rightsquigarrow 0 - i$

General Formula

Say $z = re^{i\theta}$.

The n distinct n^{th} roots of z, i.e. solutions w of eqn. $w^n = z$ are:

$$w_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \qquad k = 0, 1, 2, \ldots, n-1$$

Cartesian:

$$w_k = r^{\frac{1}{n}} \cos\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right) + r^{\frac{1}{n}} \sin\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right) i$$

Examples: (a) $z^4 = 16$ (b) 3^{rd} roots of $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

Solution:

(a) $16 = 16e^{i0}$, $\theta = 0$, $n = 4$ $\quad w_k = 16^{\frac{1}{4}} e^{i\left(\frac{0}{4} + k\frac{2\pi}{4}\right)} \rightsquigarrow 2e^{i\left(k\frac{\pi}{2}\right)}$

$w_0 = 2$, $w_1 = 2e^{i\frac{\pi}{2}} = 2i$, $w_2 = 2e^{i\pi} = -2$, $w_3 = 2e^{i\frac{3\pi}{2}} = -2i$

(b) 

$\theta = \frac{5\pi}{6}$, $n = 3$ $\rightsquigarrow$ $w_k = 1^{\frac{1}{3}} e^{i\left(\frac{5\pi}{18} + k\frac{2\pi}{3}\right)}$

$w_0 = e^{i\frac{5\pi}{18}}$, $w_1 = e^{i\frac{17\pi}{18}}$, $w_3 = e^{i\frac{29\pi}{18}}$