W16 - Notes

Complex exponential

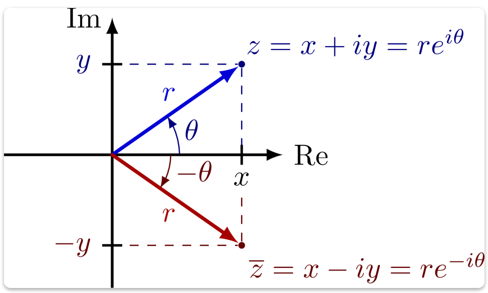
Videos

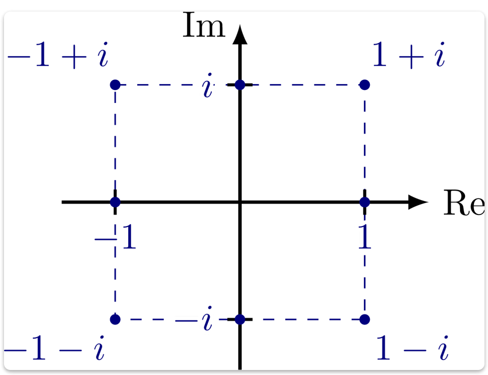
Videos, Khan Academy

- Complex exponential form

01 Theory - Complex plane and polar data

A complex number $z = x + iy$ can be represented in the plane as the point with Cartesian coordinates (x, y). The coefficient of "i" determines the vertical coordinate, and the coefficient of "1" determines the horizontal coordinate.





Let us be given a complex number $z = a + bi$.

The "real part" and "imaginary part" of z can be extracted with designated functions:

$$\mathrm{Re}(z) = a, \qquad \mathrm{Im}(z) = b, \qquad \text{for } z = a + bi$$

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i$$

The polar data (radius and angle) have special names and notations for complex numbers:

$$r = \sqrt{a^2+b^2} \qquad \theta = \tan^{-1}(b/a) \overset{?}{+} \pi$$

$$= |z| \qquad = \mathrm{Arg}(z)$$

$$= \text{“modulus” of } z \qquad = \text{“argument” of } z$$

Using this notation, we see that product with the conjugate gives square of modulus:

$$z\bar{z} \;=\; |z|^2$$

02 Theory - cis, Euler, products, powers

Multiplication of complex numbers is much easier to understand when the numbers are written using polar form.

There is a shorthand ‘cis’ notation. Convert to polar coordinates, so $a = r\cos\theta$ and $b = r\sin\theta$:

$$\begin{aligned} a+bi \quad &\gg\!\!\gg \quad r\cos\theta + r\sin\theta\, i \\ &\gg\!\!\gg \quad r\big(\cos\theta + i\sin\theta\big) \\ &\gg\!\!\gg \quad r\,\mathrm{cis}\,\theta \end{aligned}$$

The “cis” stands for $\cos\theta + i\sin\theta$. For example:

$$\sqrt{2} - \sqrt{2}i \quad \gg\!\!\gg \quad 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\gg\!\!\gg \quad 2\cos\left(-\frac{\pi}{4}\right) + 2\sin\left(-\frac{\pi}{4}\right)i$$

$$\gg\!\!\gg \quad 2\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$$

Euler Formula

General **Euler Formula**:

$$re^{i\theta} \;=\; r\cos\theta + i\,r\sin\theta$$

On the unit circle $r = 1$:

$$e^{i\theta} \;=\; \cos\theta + i\sin\theta$$

The form $re^{i\theta}$ expresses the *same data* as the cis form. The principal advantage of the form $re^{i\theta}$ is that it *reveals the rule for multiplication*, which comes from exponent laws:

Complex multiplication - Exponential form

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} \quad = \quad (r_1r_2)\,e^{i(\theta_1+\theta_2)}$$

In words:

- Multiply radii
- Add angles

Notice:

$$\text{multiply by } e^{i\frac{\pi}{2}} \qquad \Longleftrightarrow \qquad \text{rotate by } +90^\circ$$

Notice:

$$e^{i\frac{\pi}{2}} = +i$$

Therefore i ‘acts upon’ other numbers by rotating them 90° counterclockwise!

De Moivre’s Theorem - Complex powers

In exponential notation:

$$\left(re^{i\theta}\right)^n \;=\; r^n e^{i\cdot n\theta}$$

In cis notation:

$$\left(r \operatorname{cis}\theta\right)^n \;=\; r^n \operatorname{cis}(n\theta)$$

Expanded cis notation:

$$\left(r\cos\theta + i\, r\sin\theta\right)^n \;=\; r^n\cos(n\theta) + i\, r^n \sin(n\theta)$$

So the power of n acts like this:

- **Stretch:** r to r^n
- **Rotate:** by n increments of θ

Extra - Derivation of Euler Formula

Recall the power series for e^x:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \qquad = \sum_{i=0}^{\infty} \frac{1}{i!}x^i$$

Plug in $x = i\theta$:

$$e^{i\theta} \quad \ggg \quad 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots +$$

Simplify terms:

$$\ggg \quad 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 - \frac{1}{6!}\theta^6 - \frac{1}{7!}i\theta^7 + \frac{1}{8!}\theta^8 + \cdots$$

Separate by i-factor. Select out the $\overbrace{\text{terms with } i}$:

$$\gg\gg \quad 1 \overbrace{+i\theta} - \frac{1}{2!}\theta^2 \overbrace{-\frac{1}{3!}i\theta^3} + \frac{1}{4!}\theta^4 \overbrace{+\frac{1}{5!}i\theta^5} - \frac{1}{6!}\theta^6 \overbrace{-\frac{1}{7!}i\theta^7} + \frac{1}{8!}\theta^8 + \cdots$$

Separate into a series without i and a series with i:

$$\gg\gg \quad \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + \left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)i$$

Identify $\cos\theta$ and $\sin\theta\, i$. Write trig series:

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots$$

Therefore $e^{i\theta} = \cos\theta + i\sin\theta$.

03 Illustration

☰ Example - Complex product, quotient, power using Euler

Define:

$$z = 2e^{i\frac{\pi}{2}} \qquad w = 5e^{i\frac{\pi}{3}}$$

Product zw:

$$zw \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right) \cdot \left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \quad (2\cdot 5)\left(e^{i\frac{\pi}{2}}\right)\left(e^{i\frac{\pi}{3}}\right) \quad \gg\gg \quad 10e^{i\frac{\pi}{2}+i\frac{\pi}{3}} \quad \gg\gg \quad \boxed{10e^{i\frac{5\pi}{6}}}$$

Quotient z/w:

$$z/w \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right) \Big/ \left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \frac{2e^{i\frac{\pi}{2}}}{5e^{i\frac{\pi}{3}}} \quad \gg\gg \quad \frac{2}{5}e^{i\frac{\pi}{2}}e^{-i\frac{\pi}{3}} \quad \gg\gg \quad \boxed{\frac{2}{5}e^{i\frac{\pi}{6}}}$$

Power z^8:

$$z^8 \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right)^8$$

$$\gg\gg \quad 2^8\left(e^{i\frac{\pi}{2}}\right)^8 \quad \gg\gg \quad 512e^{i\cdot 4\pi}$$

Notice:

$$e^{i\cdot 4\pi} \quad \gg\gg \quad \left(e^{2\pi i}\right)^2 \quad \gg\gg \quad 1^2 \quad \gg\gg \quad 1$$

Simplify:

$$512e^{i\cdot 4\pi} \quad \gg\gg \quad \boxed{512}$$

Example - Complex power from Cartesian

Compute $(3+3i)^4$.

Solution

First convert to exponential form:

$$3+3i \quad \gg\gg \quad 3\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 3\sqrt{2}e^{i\frac{\pi}{4}}$$

Compute the power:

$$(3+3i)^4 \quad \gg\gg \quad \left(3\sqrt{2}e^{i\frac{\pi}{4}}\right)^4$$

$$\gg\gg \quad 324e^{i\pi} \quad \gg\gg \quad -324$$

Complex roots

Videos

Videos, Trefor Bazett

- Finding cube roots: Find cube roots of -1

Videos, Brain Gainz

- Finding nth roots: Fourth roots of $\sqrt{3}-i$ and cube roots of -8

04 Theory - Roots formula

The exponential notation leads to a formula for a complex n^{th} root of any complex number:

$$\sqrt[n]{re^{i\theta}} \quad = \quad \sqrt[n]{r}\, e^{i\frac{\theta}{n}}$$

n distinct roots

Every complex number actually has n distinct n^{th} roots!

That’s two square roots, three cube roots, four 4^{th} roots, etc.

All complex roots

The complex roots of $z=re^{i\theta}$ are given by this formula:

$$w_k \;=\; \sqrt[n]{r}\cdot e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)} \qquad \text{for each } k=0,\,1,\,2,\,\ldots,\,n-1$$

In Cartesian notation:

$$w_k \;=\; \sqrt[n]{r}\,\cos\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)+\sqrt[n]{r}\,\sin\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)i$$

In words:

- Start with the basic root: $\sqrt[n]{r}\cdot e^{i\frac{\theta}{n}}$
- Rotate by increments of $\frac{2\pi}{n}$ to get all other roots
    - After n distinct roots, this process repeats itself

Extra - Complex roots proof

We must verify that $w_k^n = re^{i\theta}$:

$$\left(\sqrt[n]{r}\cdot e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)}\right)^n \quad \gg\gg \quad r^{\frac{n}{n}}\cdot e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)n}$$

$$\gg\gg \quad r\cdot e^{i\,(\theta+2\pi k)} \quad \gg\gg \quad re^{i\theta}e^{i\,2\pi k} \quad \gg\gg \quad re^{i\theta}$$

05 Illustration

Example - Finding all 4^{th} roots of 16

Compute all the 4^{th} roots of 16.

Solution

Write $16 = 16e^{0i}$.

Evaluate roots formula:

$$\left(16e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = 16^{\frac{1}{4}}e^{i\left(\frac{0}{4}+k\frac{2\pi}{4}\right)}$$

Simplify:

$$\gg\gg \quad 2e^{i\cdot k\frac{\pi}{2}} \quad \gg\gg \quad 2,\; 2i,\; -2,\; -2i$$

Example - Finding 2^{nd} roots of $2i$

Find both 2^{nd} roots of $2i$.

Solution

Write $2i = 2e^{i\frac{\pi}{2}}$.

Evaluate roots formula:

$$\left(2e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} \quad \ggg \quad w_k = \sqrt{2}e^{i\left(\frac{\pi/2}{2} + k\frac{2\pi}{2}\right)}$$

$$\ggg \quad \sqrt{2}e^{i\left(\frac{\pi}{4} + k\pi\right)}$$

Compute the options: $k = 0,\, 1$:

$$\ggg \quad \sqrt{2}e^{i\frac{\pi}{4}},\ \sqrt{2}e^{i\frac{5\pi}{4}}$$

Convert to rectangular:

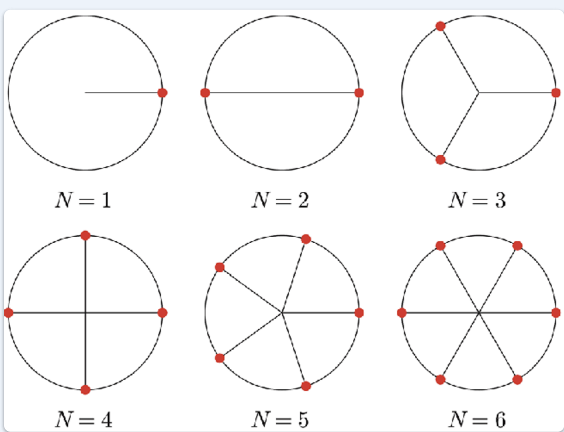
$$\ggg \quad \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\ggg \quad 1+i,\ 1-i$$

Example - Some roots of unity

Find the 1^{st} and 2^{nd} and 3^{rd} and 4^{th} and 5^{th} and 6^{th} roots of the number 1.

Solution



(1) 1^{st}

Write $1 = e^{0i}$. Evaluate roots formula. There is no possible k:

$$\left(e^{0i}\right)^{\frac{1}{1}} \quad \ggg \quad e^{0i} \quad \ggg \quad 1$$

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(2) 2^{nd}

Write $1 = e^{0i}$. Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{2}} \quad \ggg \quad w_k = e^{i\left(\frac{0}{2} + k\frac{2\pi}{2}\right)} \qquad k = 0,\, 1$$

Compute the two options, $k = 0,\, 1$:

$$\gg\gg \quad 1,\ e^{\pi i} \quad \gg\gg \quad 1,\ -1$$

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(3) 3rd

Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{3}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{3}+k\frac{2\pi}{3}\right)}$$

Compute the options: $k = 0,\ 1,\ 2$:

$$\gg\gg \quad 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}} \quad \gg\gg \quad 1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

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(4) 4th

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{4}+k\frac{2\pi}{4}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3$:

$$1,\ e^{i\frac{\pi}{2}},\ e^{i\pi},\ e^{i\frac{3\pi}{2}} \quad \gg\gg \quad 1,\ i,\ -1,\ -i$$

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(5) 5th

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{5}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{5}+k\frac{2\pi}{5}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4$:

$$1,\ e^{i\frac{2\pi}{5}},\ e^{i\frac{4\pi}{5}},\ e^{i\frac{6\pi}{5}},\ e^{i\frac{8\pi}{5}}$$

Don't simplify, it's not feasible.

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(6) 6th

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{6}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{6}+k\frac{2\pi}{6}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4,\ 5$:

$$1,\ e^{i\frac{2\pi}{6}},\ e^{i\frac{4\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{8\pi}{6}},\ e^{i\frac{10\pi}{6}}$$

Simplify:

$$\gg\gg \quad 1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$$