

W01 Notes

Volume using cylindrical shells

Videos

Review

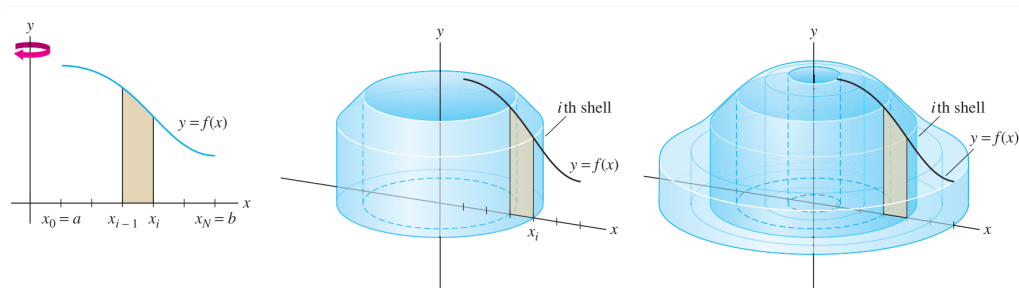
- [Volume using cross-sectional area](#)
- [Disk/washer method - 01](#)
- [Disk/washer method - 02](#)
- [Disk/washer method - 03](#)

Shells

- [Shell method - 01](#)
- [Shell method - 02](#)
- [Shell method - 03](#)

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy -plane. Rotate this about the y -axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi R h \Delta r$:

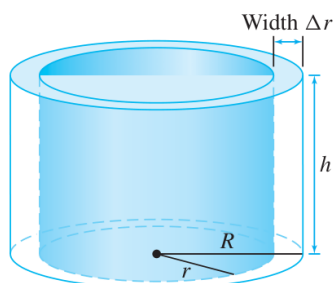


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi R h \Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \rightarrow dr$ and the number of shells becomes infinite, their total volume is given by an integral.

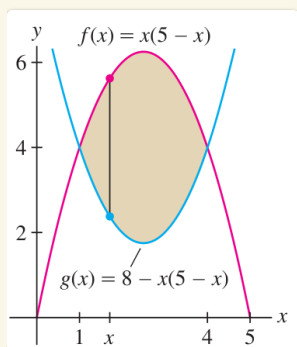
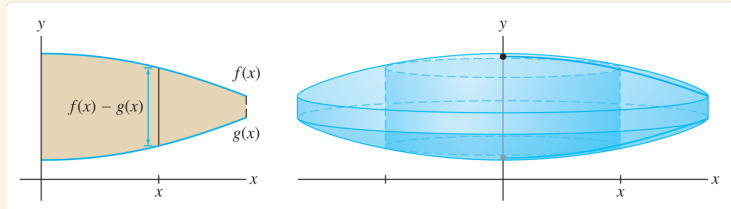
Volume by shells - general formula

$$V = \int_a^b 2\pi R h \, dr$$

In any concrete volume calculation, we simply interpret each factor, ' R ' and ' h ' and ' dr ', and determine a and b in terms of the variable of integration that is set for r .

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?



02 Illustration

Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y -axis. Find the volume of this 3D body.

Solution

(1) Define the cross section region.

Bounded above-right by $y = 6 - x$.

Bounded below-right by $y = 3x + 2$.

⚠ These intersect at $x = 1$.

Bounded at left by $x = 0$.

(2) Define range of integration variable.

Rotated around y -axis, therefore use x for integration variable (shells!).

Integral over $x \in [0, 1]$:

$$V = \int_0^1 2\pi R h \, dr$$

(3) Interpret R .

Radius of shell-cylinder equals distance along x :

$$R(x) = x$$

(4) Interpret h .

Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6 - x) - (3x + 2) \\ &= 4 - 4x \end{aligned}$$

(5) Interpret dr .

dr is limit of Δr which equals Δx here so $dr = dx$.

(6) Plug data in volume formula.

Insert data and compute integral:

$$\begin{aligned} V &= \int_0^1 2\pi R h \, dr \\ &= \int_0^1 2\pi \cdot x(4 - 4x) \, dx \\ &= 2\pi \left(2x^2 - \frac{4x^3}{3} \right) \Big|_0^1 = \frac{4\pi}{3} \end{aligned}$$

Exercise - Revolution of a sinusoid

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y -axis. Set up an integral that gives the volume of this region using the method of shells.

[Solution](#)

Integration by substitution

Videos

[Note: this section is non-examinable. It is included for comparison to IBP.]

- [Integration by Substitution 1](#): $\int \frac{-x}{(x+1)\sqrt{x+1}} \, dx$
- [Integration by Substitution 2](#): $\int \frac{x^5}{(1-x^3)^3} \, dx$

- [Integration by Substitution 3](#): $\int_0^1 x^2(1+x)^4 dx$
- [Integration by Substitution 4](#): $\int \frac{2x+3}{\sqrt{2x+1}} dx$
- [Integration by Substitution 5](#): $\int \frac{\sin x}{\cos^3 x} dx$
- [Integration by Substitution](#): Definite integrals, various examples

03 Theory

The method of ***u*-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

📦 Substitution

Suppose the integral has this format, for some functions f and u :

$$\int f(u(x)) \cdot u'(x) dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x) dx \gg \gg \int f(u) du$$

The technique of u -substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

Here we let $F' = f$. Thus $\int f(x) dx = F(x) + C$ for some C .

Now, if we *integrate both sides* of this equation, we find:

$$F(u(x)) = \int f(u(x)) \cdot u'(x) dx$$

And of course $F(u) = \int f(u) du - C$.

📖 Extra - Full explanation of u -substitution

(1) The substitution method comes from the **chain rule for derivatives**. The rule simply comes from *integrating on both sides* of the chain rule.

Setup: functions $F' = f$ and $u(x)$.

Let F and f be any functions satisfying $F' = f$, so F is an antiderivative of f .

Let u be another *function* and take x for its independent variable, so we can write $u(x)$.

(2) The chain rule for derivatives.

Using primes notation:

$$(F \circ u)' = (F' \circ u) \cdot u'$$

Using differentials in variables:

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

(3)

⚠ Integrate both sides of chain rule.

Integrate with respect to x :

$$\begin{aligned} \frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x) & \quad \xrightarrow{\int} \quad \int \frac{d}{dx} F(u(x)) = \int f(u(x)) \cdot u'(x) \\ & \quad \xrightarrow{\text{FTC}} \quad F(u(x)) = \int f(u(x)) \cdot u'(x) \end{aligned}$$

(4) Introduce ‘variable’ u from the u -format of the integral.

Treating u as a variable, the definition of F gives:

$$F(u) = \int f(u) du + C$$

Set the ‘variable’ u to the ‘function’ u output:

$$F(u) \Big|_{u=u(x)} = F(u(x))$$

Combining these:

$$\begin{aligned} F(u(x)) &= F(u) \Big|_{u=u(x)} \\ &= \int f(u) du \Big|_{u=u(x)} + C \end{aligned}$$

(5) Substitute for $F(u(x))$ in the integrated chain rule.

Reverse the equality and plug in:

$$\int f(u(x)) \cdot u'(x) dx = F(u(x)) = \int f(u) du \Big|_{u=u(x)} + C$$

This is “ u -substitution” in final form.

Integration by parts

Videos

Videos:

- [Integration by Parts 1](#): $\int e^x dx$ and $\int \ln x dx$
- [Integration by Parts 2](#): $\int \tan^{-1} x dx$ and $\int x \sec x dx$
- [Integration by Parts 3](#): Definite integrals
- Example: $\int e^{3x} \cos 4x dx$, two methods:
 - [Double IBP](#)
 - [Fast Solution](#)
- [Integration by Parts 6](#): $\int \sec^5 x dx$

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v :

$$\int u \cdot v' dx$$

Then the rule says we may convert the integral like this:

$$\int u \cdot v' dx \ggg u \cdot v - \int u' \cdot v dx$$

This technique comes from the **product rule for derivatives**:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u \cdot v = \int u' \cdot v dx + \int u \cdot v' dx$$

and the IBP rule follows by algebra.

Extra - Full explanation of integration by parts

(1) Setup: functions u and v' are established.

Recognize functions $u(x)$ and $v'(x)$ in the integrand:

$$\int u \cdot v' dx$$

(2) Product rule for derivatives.

Using primes notation:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

(3)

Integrate both sides of product rule.

Integrate with respect to an input variable labeled 'x':

$$(u \cdot v)' = u' \cdot v + u \cdot v' \ggg \int (u \cdot v)' dx = \int u' \cdot v dx + \int u \cdot v' dx$$

$$\stackrel{\text{FTC}}{\ggg} u \cdot v = \int u' \cdot v dx + \int u \cdot v' dx$$

Rearrange with algebra:

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v$$

(4) This is “integration by parts” in final form.

Addendum: *definite* integration by parts

Definite version of FTC.

Apply FTC to $u \cdot v$:

$$\int_a^b (u \cdot v)' dx = u \cdot v \Big|_a^b$$

(5) Integrate the derivative product rule using specified bounds.

Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v' dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

🔗 Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v ?

Here is a trick: the acronym “LIATE” spells out the order of choices – to the left for u and to the right for v :

LIATE :

$u \leftarrow \text{Logarithmic} - \text{Inverse_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$

05 Illustration

≡ Example - A and T factors

Compute the integral: $\int x \cos x dx$

Solution

(1) Choose $u = x$.

Set $u(x) = x$ because x *simplifies* when differentiated.

(By the trick: x is *Algebraic*, i.e. more “ u ”, and $\cos x$ is *Trig*, more “ v ”.)

Remaining factor must be v' :

$$v'(x) = \cos x$$

(2) Compute u' and v .

Derive u :

$$u' = 1$$

Antiderive v' :

$$v = \sin x$$

Obtain chart:

$$\begin{array}{l|l} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \longrightarrow \begin{array}{l} \int u \cdot v' \\ \int u' \cdot v \end{array} \begin{array}{l} \text{original} \\ \text{final} \end{array}$$

(3) Plug into IBP formula.

Plug in all data:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

Compute integral on RHS:

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

Note: the *point* of IBP is that this integral is easier than the first one!

(4) Final answer is: $x \sin x + \cos x + C$

Exercise - Hidden A

Compute the integral:

$$\int \ln x \, dx$$

[Solution](#)