

Calculus II - Lecture notes - W05

Hydrostatic force

Videos

Videos, Organic Chemistry Tutor

- [Hydrostatic pressure problems](#)

01 Theory

The pressure in a liquid is a function of the depth alone. This is a fundamental fact about liquids.

Pressure function

The fluid **pressure** in a liquid is a function of depth h :

$$p(h) = \rho gh$$

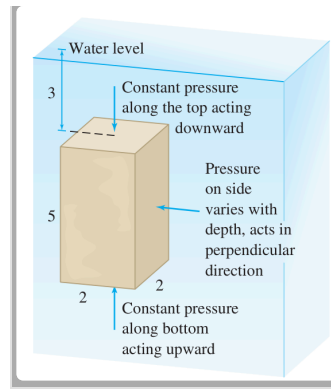
Constants:

- ρ = fluid density
- g = gravity constant

In SI units:

- $\rho = 1000\text{kg/m}^3$
- $g = 9.8\text{m/s}^2$

The pressure of a fluid acts upon any surface in the fluid by exerting a force perpendicular to the surface. Force is pressure times area. If the pressure varies across the surface, the total force must be calculated using an integral to add up differing contributions of force on each portion of the surface.



🏠 Fluid force on submerged plate

Total fluid force on plate:

$$F = \rho g \int_a^b h(y) w(y) dy$$

- $h(y)$ = depth of horizontal slice
- $w(y)$ = width of the slice
- a, b = vertical limits of surface

Use $y = a$ for top of plate (shallow edge) and $y = b$ for bottom of plate (deep edge).

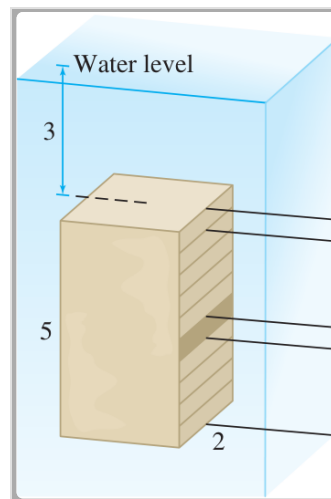
Use $h(y) = y$ when $y = 0$ at the water line, and y increases with depth.

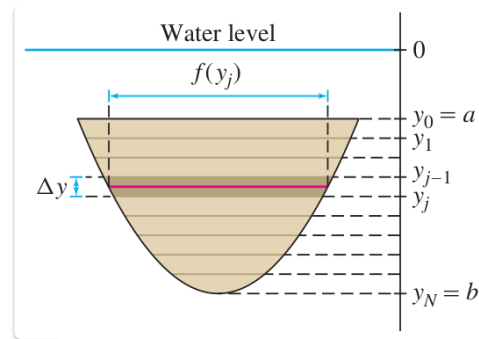
(Other $h(y)$ are possible, e.g. if)

⚠️ Vertical plate

This formula assumes the plate is *oriented straight vertically*, not slanting.

(Add the factor $\csc \theta$ for a plate tilted by angle θ .)

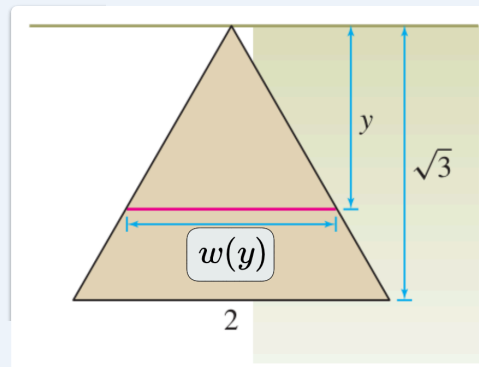




02 Illustration

≡ Example - Fluid force on a triangular plate

Find the total force on the submerged *vertical* plate with the following shape: Equilateral triangle, sides 2m, top vertex at the surface, liquid is oil with density $\rho = 900\text{kg/m}^3$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the vertex), and y increases going *down*.

Method 1: Geometry of similar triangles

Top triangle with base at $w(y)$ is similar to *total* triangle with base $\sqrt{3}$.

Therefore, corresponding parts have the same ratios.

Therefore:

$$\frac{w(y)}{y} = \frac{2}{\sqrt{3}}$$

$$\gg \gg w(y) = \frac{2}{\sqrt{3}}y$$

Method 2: Quick linear interpolation function

$$w(y) = 0 + \frac{2-0}{\sqrt{3}} \cdot y$$

$$\ggg w(y) = \frac{2}{\sqrt{3}}y$$

☰ Generalized “quick linear interpolation function” >

Generalization:

$$w(y) = \text{Edge}_1 + \frac{\text{Edge}_2 - \text{Edge}_1}{\text{Height}} (\pm y \pm a)$$

where:

- $\pm y$ is $+y$ when Edge_1 comes earlier (smaller y), and $-y$ if it comes later
- $\pm a$ is created to force the quantity $(\pm y \pm a)$ to equal 0 for the given y value at Edge_1

(2) Compute integral using width function:

Bounds:

- $y = 0$ (shallow)
- $y = \sqrt{3}$ (deep)

Integral formula:

$$F = \rho g \int_0^{\sqrt{3}} y w(y) dy$$

$$\ggg 900 \cdot 9.8 \int_0^{\sqrt{3}} y \cdot 2y/\sqrt{3} dy$$

$$\ggg 10184.5 \int_0^{\sqrt{3}} y^2 dy$$

$$\ggg 10184.5 \frac{y^3}{3} \Big|_0^{\sqrt{3}} \ggg 17640$$

03 Theory

What if the submerged surface is *not* oriented straight vertically?

The amount of surface for a horizontal strip at a given depth will be increased by a factor of $\csc \theta$ where θ is the angle of incline of the surface (with $\theta = 0$ corresponding to horizontal and

$\theta = \pi/2$ to vertical). Thus:

$$dz = \csc \theta dy$$

where dz is the thickness of a strip.

So the total force formula becomes:

🏠 Fluid force for tilted surface

Total fluid force on tilted plate:

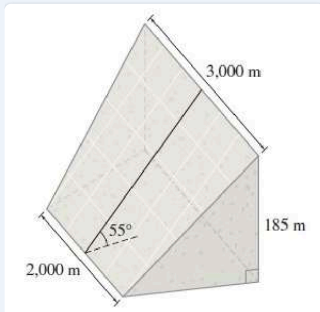
$$F = \rho g \int_a^b h w dz = \rho g \int_a^b h(y) w(y) \csc \theta dy$$

As before, y measures depth with $y = 0$ at the surface.

04 Illustration

≡ Example - Weight of water on a dam

Find the total hydrostatic force on an angled dam with the following geometric description: Tilted trapezoid. Base = 2,000m, Top = 3,000m, and *vertical* height 185m. The base is tilted at an angle of $\theta = 55^\circ$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the top edge), and increases going *down*.

“Quick linear interpolation function”:

$$w(y) = 3000 + \frac{2000 - 3000}{185}y$$

$$\gg \gg 3000 - \frac{1000}{185}y$$

(2) Incorporate angle of incline in strip thickness:

$$dz = \csc 55^\circ dy$$

So the area of a strip is:

$$dA = w(y) dz$$

$$\ggg \left(3000 - \frac{1000}{185} y \right) \csc 55^\circ dy$$

(3) Compute total force using integral formula.

Plug data into formula:

$$F = \rho g \int_a^b h(y) w(y) dz$$

$$\ggg \rho g \csc 55^\circ \int_0^{185} y \left(3,000 - \frac{1,000}{185} y \right) dy$$

$$\ggg 4.777 \times 10^{11}$$

Work

Videos

Videos, BlackPenRedPen:

- [Work performed](#): pumping water from trough
- [Work performed](#): pumping water from rectangular tank
- [Work performed](#): pumping water from conical tank
- [Work performed](#): pumping water from spherical tank

01 Theory

Work is a measure of energy expended to achieve some effect. According to physics:

$$\text{Work} = \text{Force} \times \text{Distance}$$

$$W = \int_a^b F(x) dx = \int_c^d x dF$$



To compute the work performed against gravity while *lifting some matter*, decompose the matter into *horizontal layers* at height y and thickness dy . Each layer is lifted some distance. The weight of the layer gives the force applied.

The work performed on each single layer is summed by an integral to determine the total work performed to lift all the layers:

Work performed

Work to lift a layer = $g \times \text{density} \times A(y) \times \text{height raised} \times dy$

$$\text{Total work} = \int_a^b \rho g h(y) A(y) dy$$

- $A(y)$ = area of layer
- $h(y)$ = height layer is lifted
- ρ = mass density, 1000 kg/m^3 for water
- $g = 9.8 \text{ m/s}^2$ = constant of gravitational acceleration

$$\begin{aligned} dV &= A(y) dy && \text{volume of slice} \\ dF &= \rho g dV = \rho g A dy && \text{weight/force on slice} \\ dW &= h(y) dF = \rho g h A dy && \text{work to pump out slice} \\ W &= \int dW = \int \rho g h A dy && \text{total work} \end{aligned}$$

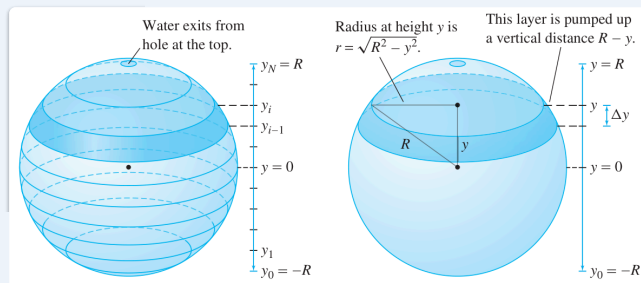
02 Illustration

Example - Pumping water from spherical tank

Calculate the work done pumping water out of a spherical tank of radius $R = 5 \text{ m}$.

Solution

(1) Divide the sphere of water into horizontal layers.



Coordinate y is $y = 0$ at the center of the sphere.

(2) Work done pumping out water is constant across any single layer

Find formula for weight of a single layer.

Area of the layer at y is $A(y) = \pi(5^2 - y^2)$ because its radius is $r = \sqrt{5^2 - y^2}$.

Volume of the layer at y is then $\pi(5^2 - y^2) dy$. $= dV$

Weight of the layer is then ~~$\rho g A dy$~~ $= g \cdot \rho \cdot \pi(5^2 - y^2) dy$. $= dF$

Plug in:

$$\rho g dV =$$

$$g = 9.8 \frac{m}{s^2}, \quad \rho = 1000 \frac{kg}{m^3} \quad \gg \gg \quad F(y) dy = \left(9800 \frac{kg}{m^2 s^2} \right) \pi (5^2 - y^2) dy$$



(3) Find formula for vertical distance a given layer is lifted.

Layer at y must be lifted by $5 - y$ to the top of the tank.

(4) Work per layer is the product.

Product of weight times height lifted:

$$\left(9800 \frac{kg}{m^2 s^2} \right) \pi (5^2 - y^2) \overbrace{(5 - y)}^{h(y)} dy$$

(5) Total work done is the integral.

Integrate over the layers:

$$W = \int_{-5}^{+5} \left(9800 \frac{kg}{m^2 s^2} \right) \pi (5^2 - y^2) (5 - y) dy \approx 2.6 \times 10^7 \text{ J}$$

(6)

☰ Extra question: what if the spigot sits 2m above the tank?

Increase the height function from $5 - y$ to $7 - y$.

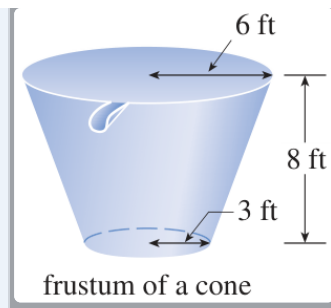
(7)

☰ Extra question: what if the tank starts at just 3m of water depth?

Integrate the water layers only: change bounds to $\int_{-5}^{-2}$.

☰ Example - Water pumped from a frustum

Find the work required to pump water out of the frustum in the figure. Assume the weight of water is $\rho = 62.5 \text{ lb/ft}^3$.

**Solution**

(1) Find weight of a horizontal slice.

Coordinate $y = 0$ at top, increasing downwards.

Use $r(y)$ for radius of cross-section circle.

Linear decrease in r from $r(0) = 6$ to $r(8) = 3$:

$$r(y) = 6 - \frac{3}{8}y$$

Area is πr^2 :

$$\text{Area}(y) = \pi \left(6 - \frac{3}{8}y\right)^2$$

Weight = density $\times$ area $\times$ thickness:

$$\text{weight of layer} = \rho \pi \left(6 - \frac{3}{8}y\right)^2 dy$$

(2) Find work to pump out a horizontal layer.

Layer at y is raised a distance of y .

Work to raise layer at y :

$$\rho \pi y \left(6 - \frac{3}{8}y\right)^2 dy$$

(3) Integrate over all layers.

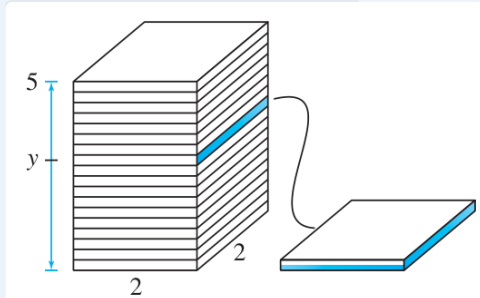
Integrate from top to bottom of frustum:

$$\begin{aligned} \int_0^8 \rho \pi y \left(6 - \frac{3}{8}y\right)^2 dy &= 528\pi\rho \\ &= 528\pi \cdot 62.5 \\ &\approx 1.04 \times 10^5 \text{ ft-lb} \end{aligned}$$

Final answer is 1.04×10^5 ft-lb.

≡ Example - Raising a building

Find the work done to raise a cement columnar building of height 5 m and square base 2 m per side. Cement has a density of 1500 kg/m^3 .



Solution

(1) Divide the building into horizontal layers.

Work done raising up the layers is constant for each layer.

(2) Find formula for weight of each layer.

$$\text{Volume} = \text{area} \times \text{thickness} = 4 \, dy$$

$$\text{Mass} = \text{density} \times \text{volume} = 1500 \times 4 \, dy = 6000 \, dy$$

$$\text{Weight of layer} = g \times \text{mass} = 9.8 \times 6000 \, dy = 58800 \, dy$$

(3) Find formula for work performed lifting one layer into place.

$$\text{Work} = \text{weight} \times \text{distance lifted} = 58800 \times y \, dy$$

(4) Find total work as integral over the layers.

$$\text{Total work} = \int_0^5 58800 y \, dy = 735 \text{ kJ}$$

≡ Example - Raising a chain

An 80 ft chain is suspended from the top of a building. Suppose the chain has weight density 0.5 lb/ft . What is the total work required to reel in the chain?

Solution

(1) Divide the chain into horizontal layers.

Each layer has vertical thickness dy .

Each layer has weight $0.5 \, dy$ in lb.

(2) Find formula for distance each layer is raised.

Each layer is raised from y to 80 ft, a distance of $(80 - y)$ ft.

(3) Compute total work.

Work to raise each layer is weight times distance raised:

$$\text{Work to raise layer} = (80 - y) \cdot 0.5 \, dy$$

Add the work over all layers:

$$\int_0^{80} (80 - y) \cdot 0.5 \, dy = 1600 \text{ ft-lb}$$

Example - Raising a leaky bucket

Suppose a bucket is hoisted by a cable up an 80 ft tower. The bucket is lifted at a constant rate of 2 ft/sec and is leaking water weight at a constant rate of 0.2 lb/sec. The initial weight of water is 50 lb. What is the total work performed against gravity in lifting the water? (Ignore the bucket itself and the cable.)

Solution

(1) Convert to static format.

Compute rate of water weight loss per unit of vertical height:

$$\frac{\text{rate of leak}}{\text{rate of lift}} = \text{leaked weight per foot}, \quad \frac{0.2 \text{ lb/sec}}{2 \text{ ft/sec}} = 0.1 \text{ lb/ft}$$

Choose coordinate $y = 0$ at base, $y = 80$ at top.

Compute water weight at each height y :

$$\text{water weight} = (50 - 0.1y) \text{ lb}$$

(2)

Work formula:

Total work is integral of force times infinitesimal distance:

$$\text{work} = \int_a^b F(y) dy$$

(3) Integrate weight times dy .

Plug in weight as force:

$$\text{work} = \int_0^{80} (50 - 0.1y) dy$$

Compute integral:

$$\ggg \quad 50y - 0.05y^2 \Big|_0^{80} = 3680 \text{ ft-lb}$$

 **In this example, the same bucket passes through each height, it is not sliced into layers.**

- This integral adds work done to lift matter through each dy as if in parallel, and dy is thus representing *distance lifted*.
- Previous examples have integral adding work done to lift matter through some y or $5 - y$; the matter was sliced into layers with dy featuring in the *weight* of a portion.