Calculus II - Lecture notes - W05

Hydrostatic force

Videos

Videos, Organic Chemistry Tutor

- Hydrostatic pressure problems

01 Theory

The pressure in a liquid is a function of the depth alone. This is a fundamental fact about liquids.

Pressure function

The fluid **pressure** in a liquid is a function of depth h:

$$p(h) = \rho g h$$

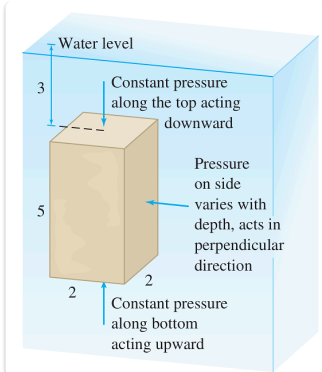
Constants:

- $\rho =$ fluid density
- $g =$ gravity constant

In SI units:

- $\rho = 1000 \mathrm{kg}/\mathrm{m}^3$
- $g = 9.8 \mathrm{m}/\mathrm{s}^2$

The pressure of a fluid acts upon any surface in the fluid by exerting a force perpendicular to the surface. Force is pressure times area. If the pressure varies across the surface, the total force must be calculated using an integral to add up differing contributions of force on each portion of the surface.



Fluid force on submerged plate

Total fluid force on plate:

$$F = \rho g \int_a^b h(y)\, w(y)\, dy$$

- $h(y) =$ depth of horizontal slice
- $w(y) =$ width of the slice
- $a,\, b =$ vertical limits of surface

Use $y = a$ for top of plate (shallow edge) and $y = b$ for bottom of plate (deep edge).

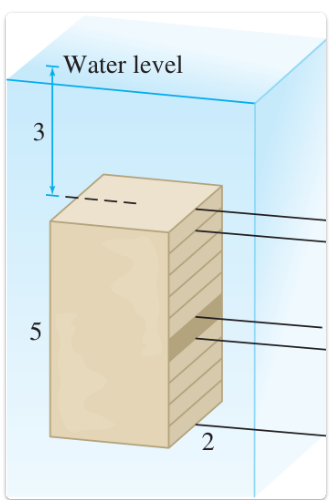
Use $h(y) = y$ when $y = 0$ at the water line, and y increases with depth.

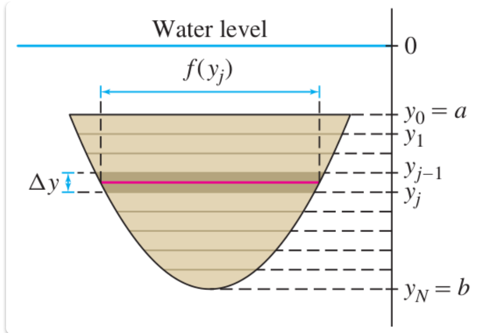
(Other $h(y)$ are possible, e.g. if)

⚠ Vertical plate

This formula assumes the plate is *oriented straight vertically*, not slanting.

(Add the factor $\csc\theta$ for a plate tilted by angle θ.)

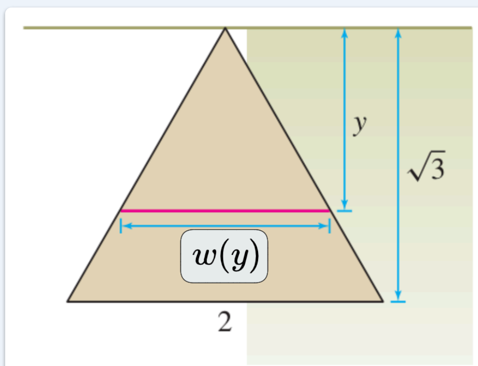




02 Illustration

☰ Example - Fluid force on a triangular plate

Find the total force on the submerged *vertical* plate with the following shape: Equilateral triangle, sides 2m, top vertex at the surface, liquid is oil with density $\rho = 900\text{kg}/\text{m}^3$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the vertex), and y increases going *down*.

Method 1: Geometry of similar triangles

Top triangle with base at $w(y)$ is similar to *total* triangle with base $\sqrt{3}$.

Therefore, corresponding parts have the same ratios.

Therefore:

$$\frac{w(y)}{y} = \frac{2}{\sqrt{3}}$$

$$\ggg \quad w(y) = \frac{2}{\sqrt{3}} y$$

Method 2: Quick linear interpolation function

$$w(y) \;=\; 0 + \frac{2-0}{\sqrt{3}} \cdot y$$

$$\gg\gg \quad w(y) \;=\; \frac{2}{\sqrt{3}} y$$

Generalized "quick linear interpolation function"

Generalization:

$$w(y) \;=\; \mathrm{Edge}_1 + \frac{\mathrm{Edge}_2 - \mathrm{Edge}_1}{\mathrm{Height}} (\pm y \pm a)$$

where:

- $\pm y$ is $+y$ when Edge_1 comes earlier (smaller y), and $-y$ if it comes later
- $\pm a$ is created to force the quantity $(\pm y \pm a)$ to equal 0 for the given y value at Edge_1

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(2) Compute integral using width function:

Bounds:

- $y = 0$ (shallow)
- $y = \sqrt{3}$ (deep)

Integral formula:

$$F \;=\; \rho g \int_0^{\sqrt{3}} y\, w(y)\, dy$$

$$\gg\gg \quad 900 \cdot 9.8 \int_0^{\sqrt{3}} y \cdot 2y/\sqrt{3}\, dy$$

$$\gg\gg \quad 10184.5 \int_0^{\sqrt{3}} y^2\, dy$$

$$\gg\gg \quad 10184.5 \frac{y^3}{3}\Big|_0^{\sqrt{3}} \quad \gg\gg \quad \boxed{17640}$$

03 Theory

What if the submerged surface is *not* oriented straight vertically?

The amount of surface for a horizontal strip at a given depth will be increased by a factor of $\csc\theta$ where θ is the angle of incline of the surface (with $\theta = 0$ corresponding to horizontal and

$\theta = \pi/2$ to vertical). Thus:

$$dz = \csc\theta\, dy$$

where dz is the thickness of a strip.

So the total force formula becomes:

Fluid force for tilted surface

Total fluid force on tilted plate:

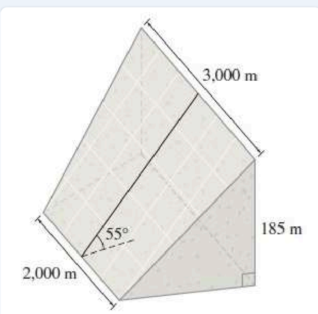
$$F \;=\; \rho g \int_a^b h\, w\, dz \;=\; \rho g \int_a^b h(y)\, w(y) \csc\theta\, dy$$

As before, y measures depth with $y = 0$ at the surface.

04 Illustration

Example - Weight of water on a dam

Find the total hydrostatic force on an angled dam with the following geometric description: Tilted trapezoid. Base $= 2,000$m, Top $= 3,000$m, and *vertical* height 185m. The base is tilted at an angle of $\theta = 55°$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the top edge), and increases going *down*.

"Quick linear interpolation function":

$$w(y) \;=\; 3000 + \frac{2000 - 3000}{185} y$$

$$\gg\gg \quad 3000 - \frac{1000}{185} y$$

(2) Incorporate angle of incline in strip thickness:

$$dz = \csc 55^\circ\, dy$$

So the area of a strip is:

$$dA \;=\; w(y)\, dz$$

$$\gg\gg \quad \left(3000 - \frac{1000}{185} y\right) \csc 55^\circ\, dy$$

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(3) Compute total force using integral formula.

Plug data into formula:

$$F \;=\; \rho g \int_a^b h(y)\, w(y)\, dz$$

$$\gg\gg \quad \rho g \csc 55^\circ \int_0^{185} y \left(3,000 - \frac{1,000}{185} y\right) dy$$

$$\gg\gg \quad \boxed{4.777 \times 10^{11}}$$

Work

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- Work performed: pumping water from trough
- Work performed: pumping water from rectangular tank
- Work performed: pumping water from conical tank
- Work performed: pumping water from spherical tank

01 Theory

Work is a measure of energy expended to achieve some effect. According to physics:

$$\text{Work} = \text{Force} \times \text{Distance}$$

$$W \;=\; \int_a^b F(x)\, dx \qquad \text{OR} \qquad W \;=\; \int_a^b x\, dF$$

To compute the work performed against gravity while *lifting some matter*, decompose the matter into *horizontal layers* at height y and thickness dy. Each layer is lifted some distance. The weight of the layer gives the force applied.

The work performed on each single layer is summed by an integral to determine the total work performed to lift all the layers:

Work performed

$$\text{Work to lift a layer} \;=\; \text{height raised} \times g \times \text{density} \times A(y) \times dy$$

$$\text{Total work} \;=\; \int_a^b \rho g\, h(y) A(y)\, dy$$

- $A(y) =$ area of layer
- $h(y) =$ height layer is lifted
- $\rho =$ mass density $= 1000\,\mathrm{kg}/\mathrm{m}^3$ for water
- $g = 9.8\,\mathrm{m}/\mathrm{s}^2 =$ constant of gravitational acceleration

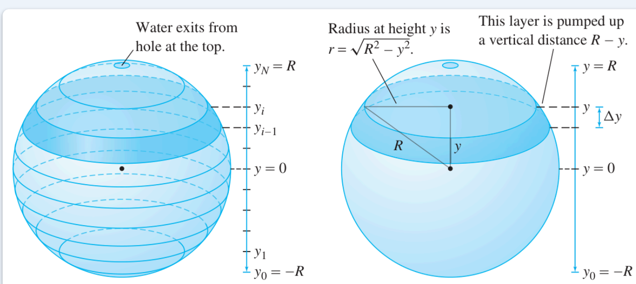
02 Illustration

Example - Pumping water from spherical tank

Calculate the work done pumping water out of a spherical tank of radius $R = 5\,\mathrm{m}$.

Solution

(1) Slice the tank of water into horizontal layers:



Coordinate y is $y = 0$ at the center of the sphere, increasing upwards.

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(2) Calculate weight of single slice:

$$\text{area of slice} \;=\; A(y) \;=\; \pi r^2 \quad \ggg \quad \pi(5^2 - y^2)$$

$$\text{volume of slice} \;=\; dV \;=\; A(y)\, dy \quad \ggg \quad \pi(5^2 - y^2)\, dy$$

$$\text{weight of slice} \;=\; dF \;=\; \rho g dV \;=\; \rho g \pi(5^2 - y^2)\, dy$$

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(3) Work to lift out single slice:

Distance to raise a slice:

$$h(y) \;=\; 5 - y$$

Then:

$$\text{work to lift out slice} \;=\; dW \;=\; h(y)dF \;=\; \rho g \pi h(y)(5^2 - y^2)\,dy$$

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(4) Total work by integrating dW over all slices:

$$\int dW \quad \ggg \quad \int_{-5}^{+5} \left(9800\,\frac{\text{kg}}{\text{m}^2\text{s}^2}\right)\pi(5^2 - y^2)(5 - y)\,dy \qquad \text{(Note A)}$$

$$\ggg \quad \approx 2.6 \times 10^7\ \text{J}$$

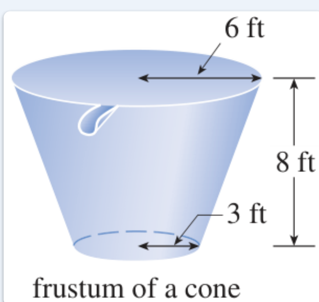
Note A: The integration runs over all slices, which start at $y = -5$ (bottom of tank), and end at $y = +5$ (top of tank).

Question: Extra question: what if the spigot sits 2m above the tank?

Question: Extra question: what if the tank starts at just 3m of water depth?

Example - Water pumped from a frustum

Find the work required to pump water out of the frustum in the figure. Assume the weight of water is $\rho = 62.5\,\text{lb}/\text{ft}^3$.



Solution

(1) Find weight of a horizontal slice.

Coordinate $y = 0$ at top, increasing downwards.

Use $r(y)$ for radius of cross-section circle.

Linear decrease in r from $r(0) = 6$ to $r(8) = 3$:

$$r(y) = 6 - \frac{3}{8}y$$

Area is πr^2:

$$\text{Area}(y) = \pi\left(6 - \tfrac{3}{8}y\right)^2$$

Weight $=$ density $\times$ area $\times$ thickness:

$$\text{weight of layer} = \rho\pi\left(6 - \tfrac{3}{8}y\right)^2 \, dy$$

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(2) Find work to pump out a horizontal layer.

Layer at y is raised a distance of y.

Work to raise layer at y:

$$\rho\pi y\left(6 - \tfrac{3}{8}y\right)^2 \, dy$$

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(3) Integrate over all layers.

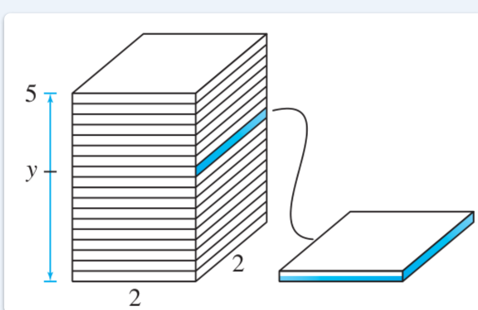
Integrate from top to bottom of frustum:

$$\begin{aligned} \int_0^8 \rho\pi y\left(6 - \tfrac{3}{8}y\right)^2 \, dy &= 528\pi\rho \\ &= 528\pi \cdot 62.5 \\ &\approx 1.04 \times 10^5 \text{ ft-lb} \end{aligned}$$

Final answer is 1.04×10^5 ft-lb.

☰ Example - Raising a building

Find the work done to raise a cement columnar building of height $5\,\text{m}$ and square base $2\,\text{m}$ per side. Cement has a density of $1500\,\text{kg}/\text{m}^3$.



Solution

(1) Weight of each layer:

$$dV = A(y)\,dy \quad \gg\!\gg \quad 4\,dy$$

$$dM = \rho dV \quad \gg\!\gg \quad 1500 \cdot 4\,dy$$

$$dF = g\,dM \quad \gg\!\gg \quad 9.8 \cdot 6000\,dy$$

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(2) Work to lift layer into place:

$$dW = \text{weight} \times \text{distance raised} \quad \gg\!\gg \quad y \cdot 58800\,dy$$

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(3) Find total work as integral over the layers:

$$W = \int dW \quad \gg\!\gg \quad \int_0^5 58800y\,dy$$

$$\gg\!\gg \quad 735\,\text{kJ}$$

☰ Example - Raising a chain

An $80\,\text{ft}$ chain is suspended from the top of a building. Suppose the chain has weight density $0.5\,\text{lb/ft}$. What is the total work required to reel in the chain?

Solution

(1) Compute weight of a 'link' (vertical slice of the chain):

$$\begin{aligned} dF &= \text{density} \times \text{length} \\ &= 0.5\,dy \end{aligned}$$

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(2) Work dW to raise link to top:

Each link (slice) is raised from height y to height 80:

$$h(y) = (80 - y)\,\text{ft}$$

Then:

$$dW = (80 - y) \cdot 0.5\,dy$$

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(3) Integrate over the chain for total work:

$$\int dW \quad \gg\!\gg \quad \int_0^{80} (80 - y) \cdot 0.5\, dy$$

$$\gg\!\gg \quad 1600\,\text{ft-lb}$$

Example - Raising a leaky bucket

Suppose a bucket is hoisted by a cable up an $80\,\text{ft}$ tower. The bucket is lifted at a constant rate of $2\,\text{ft/sec}$ and is leaking water weight at a constant rate of $0.2\,\text{lb/sec}$. The initial weight of water is $50\,\text{lb}$. What is the total work performed against gravity in lifting the water? (Ignore the bucket itself and the cable.)

Solution

(1) Compute total force from water $F(y)$:

Choose coordinate $y = 0$ at base, $y = 80$ at top.

Rate of water weight loss per unit height:

$$\frac{\text{rate of leak}}{\text{rate of lift}} \quad = \quad \text{leaked weight per foot}$$

$$\gg\!\gg \quad \frac{0.2\,\text{lb/sec}}{2\,\text{ft/sec}} \quad \gg\!\gg \quad 0.1\,\text{lb/ft}$$

Total water weight at height y:

$$F(y) \quad = \quad (50 - 0.1y)\,\text{lb}$$

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(2) Work to raise bucket by dy:

$$dW \quad = \quad F(y)\,dy \quad \gg\!\gg \quad (50 - 0.1y)\,dy$$

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(3) Total work by integrating dW:

$$W \quad = \quad \int_a^b dW \quad \gg\!\gg \quad \int_a^b (50 - 0.1y)\,dy$$

$$\gg\!\gg \qquad 50y - 0.05y^2\Big|_0^{80} = 3680\,\text{ft-lb}$$

⚠ Change of method and integral formula!

For this example, we use the formula $\int F(y)\,dy$ rather than the formula $\int h(y)\,dF$ used in the earlier examples.

- This integral sums over the work dW to lift macroscopic material through each microscopic dy as if in sequence, and dy thus represents *distance lifted*.
- Earlier examples summed over the work dW to lift microscopic material through the macroscope $h(y)$ (all the way up).