Calculus II - Lecture notes - W06

Moments and center of mass

Videos

Videos, Math Dr. Bob:

- Moments and CoM 01: Points masses on a line
- Moments and CoM 02: Points masses in the plane
- Moments and CoM 03: Planar lamina of uniform density
- Moments and CoM 04: Integral formula for planar lamina
- Moments and CoM 05: Rod of non-uniform density

03 Theory

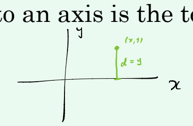
Moment

The **moment** of a region to an axis is the total (integral) of mass times distance to that axis:

Moment to x:

$\int y\, dM$

$\parallel$



$$M_x = \int \rho\, y\, dA \qquad \text{(general formula)}$$

w/out y becomes $\int \rho\, dA$

$= \int dM$

$= M$ = total mass

$$M_x = \int_c^d \rho\, y\, \big(g_2(y) - g_1(y)\big)\, dy \qquad \text{(region between functions } g_2 \text{ and } g_1)$$

Moment to y:

$$M_y = \int \rho\, x\, dA \qquad \text{(general formula)}$$

$$M_y = \int_a^b \rho\, x\, \big(f_2(x) - f_1(x)\big)\, dx \qquad \text{(region between functions } f_2 \text{ and } f_1)$$

Notice the *swap* in letters

- M_y integrand has x factor
- M_x integrand has y factor

Notice the total mass

If you remove x or y factors from the integrands, the integrals give **total mass** M.

These formulas are obtained by slicing the region into rectangular strips that are parallel to the axis in question.

The area *per strip* is then:

- $f(x)\,dx$ — region under $y = f(x)$
- $(f_2 - f_1)\,dx$ — region between f_1 and f_2
- $g(y)\,dy$ — region 'under' $x = g(y)$
- $(g_2 - g_1)\,dy$ — region between g_1 and g_2

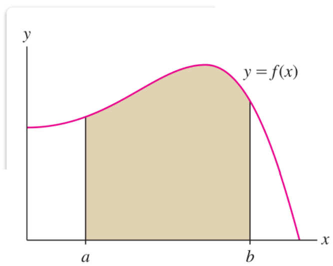


FIGURE 5 Lamina occupying the region under the graph of $f(x)$ over $[a, b]$.

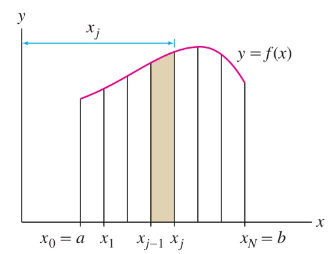
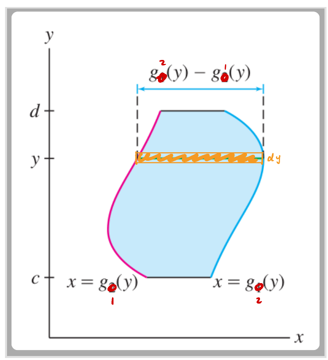


FIGURE 6 The shaded strip is nearly rectangular of area with $f(x_j)\Delta x$.



$(g_2 - g_1)\,dy = dM$

$M_x = \int \rho\, y\, dA$

$= \int_c^d \rho\, y\,(g_2 - g_1)\,dy$

The idea of moment is related to:

- Torque balance and angular inertia
- Center of mass

The **center of mass (CoM)** of a solid body is a single point with two important properties:

1. CoM = "average position" of the body
   - The average position determines an *effective center* of dynamics. For example, gravity acting on every bit of mass of a rigid body acts the same as a force on the CoM alone.
2. CoM = "balance point" of the body
   - The net *torque* (rotational force) about the CoM, generated by a force distributed over the body's mass, equivalently a force on the CoM, is zero.

Centroid

When the body has *uniform density*, then the CoM is also called the **centroid**.

Center of mass from moments

Coordinates of the CoM:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Here M is the total mass of the body.

Center of mass from moments - explanation

Notice how these formulas work. The total mass is always $M = \int \rho \, dA$. The moment to y (for example) is $M_y = \int \rho \, x \, dA$. Dividing these two values:

$$\bar{x} = \frac{M_y}{M} \quad \gg\gg \quad \frac{\int \rho \, x \, dA}{\int \rho \, dA} \quad \gg\gg \quad \frac{\int x \, dA}{\int dA} \quad \gg\gg \quad \frac{\int x \, dA}{A} = \frac{1}{A}\int x \, dA$$

where $A =$ total area.

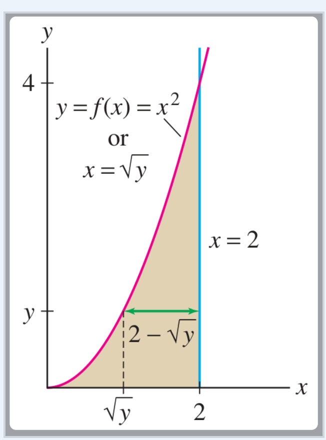
$\bar{f} = \frac{1}{A}\int f \, dA$

In other words, through the formula $\bar{x} = \frac{M_y}{M}$, we find that $\bar{x}$ is the *average value of x* over the region with area A.

04 Illustration

Example - CoM of a parabolic plat

Find the CoM of the region depicted:



Solution

$$\int \rho \, dA = \int_0^2 \rho \, (f_2 - f_1) \, dx$$

$f_2 = x^2$

$f_1 = 0$

(1) Compute the total mass:

Area under the curve with density factor ρ:

$$M = \int_0^2 \rho \, x^2 \, dx \quad \gg\gg \quad \rho \, \frac{x^3}{3}\bigg|_0^2 \quad \gg\gg \quad \frac{8\rho}{3}$$

(2) Compute M_y:

Formula:

$$M_y = \int_a^b \rho\, x\, dA$$

Interpret and calculate:

$$M_y = \int_0^2 \rho\, x f(x)\, dx \quad \gg\gg \quad \rho \int_0^2 x^3\, dx$$

$$\gg\gg \quad 4\rho = M_y$$

(3) Compute M_x:

Formula:

$$M_x = \int_c^d \rho\, y\, dA$$

$$= \int_0^2 \rho \tfrac{1}{2}(x^2 - 0^2)^2\, dx$$

$$= \rho \int_0^2 \tfrac{1}{2} x^4\, dx$$

$$= \rho \frac{x^5}{10}\Big|_0^2 = \rho \frac{32}{10} = \frac{16\rho}{5}$$

Width of horizontal strips between the curves:

$$w(y) = 2 - \sqrt{y}$$

Interpret dA:

$$dA = (2 - \sqrt{y})\, dy$$

Calculate integral:

$$M_x = \int_c^d \rho\, y\, dA \quad \gg\gg \quad \int_0^4 \rho\, y(2 - \sqrt{y})\, dy$$

$$\gg\gg \quad \int_0^4 \rho\, y(2 - \sqrt{y})\, dy \quad \gg\gg \quad \int_0^4 \rho\, 2y\, dy - \int_0^4 \rho\, y^{3/2}\, dy$$

$$\gg\gg \quad \frac{16\rho}{5} = M_x$$

(4) Compute CoM coordinates from moments:

CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Insert data:

$$\bar{x} = \frac{4\rho}{8\rho/3} \quad \gg\gg \quad \frac{3}{2}, \qquad \bar{y} = \frac{16\rho/5}{8\rho/3} \quad \gg\gg \quad \frac{6}{5}$$

$$\text{CoM} \;=\; (\bar{x}, \bar{y}) = \boxed{\left(\frac{3}{2}, \frac{6}{5}\right)}$$

05 Theory

A downside of the technique above is that to find M_x we needed to convert the region into functions in y. This would be hard to do if the region was given as the area under a curve $y = f(x)$ but $f^{-1}(y)$ cannot be found analytically. An alternative formula can help in this situation.

Midpoint of strips for opposite variables

When the region lies between $f_1(x)$ and $f_2(x)$, we can find M_x with an x-integral:

$$M_x = \int_a^b \rho \, \tfrac{1}{2} \left(f_2^2 - f_1^2\right) dx \qquad \text{(region between } f_1 \text{ and } f_2\text{)}$$

When the region lies between $g_1(y)$ and $g_2(y)$, we can find M_y with a y-integral:

$$M_y = \int_c^d \rho \, \tfrac{1}{2} \left(g_2^2 - g_1^2\right) dy \qquad \text{(region between } g_1 \text{ and } g_2\text{)}$$

Region under a curve

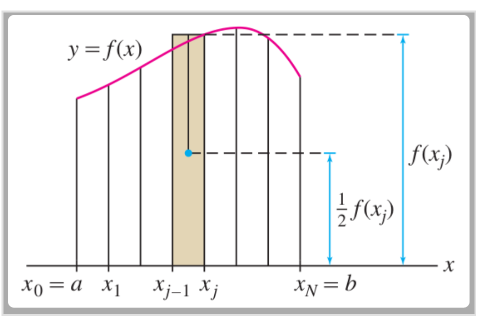
For the region "under the curve" $y = f(x)$, just set:

$$f_1 = 0, \quad f_2 = f$$

For the region "under the curve" $x = g(x)$, set:

$$g_1 = 0, \quad g_2 = g$$

The idea for these formulas is to treat each vertical strip as a point concentrated at the CoM of the vertical strip itself.



CoM of vert. strip is

$\frac{1}{2}f(x)$ ⇝ $\bar{y} = \dfrac{M_x^{strip}}{M^{strip}}$ ⇝ dM

@ $y = \frac{1}{2}f(x)$ i.e. $\frac{1}{2}(f_1(x) + f_2(x))$

$(x, \frac{1}{2}f(x))$

So $M_x^{strip} = \frac{1}{2}f(x)\,dM$

$dM = \rho f(x)\,dx$ $\quad f_2(x) - f_1(x)$

So $M_x^{strip} = \rho\frac{1}{2}f(x)^2\,dx$

So $M_x = \int_a^b \rho\frac{1}{2}f(x)^2\,dx$

$\int \rho\frac{1}{2}(f_1+f_2)(f_2-f_1)\,dx$

$\gg\gg \int \rho\frac{1}{2}(f_2^2 - f_1^2)\,dx$

The height to this midpoint is $\frac{1}{2}f(x)$, and the area of the strip is $f(x)\,dx$, so the integral becomes $\int \rho \, \frac{1}{2} f(x)^2 \, dx$.

Midpoint of strips formula - full explanation

- If the strip is located at some x, with y values from 0 up to $f(x)$, then:

$$\text{CoM of strip} = \left(x,\ \frac{1}{2}f(x)\right)$$

- The area of the strip is $dA = f(x)\,dx$. So the integral formula for M_x can be recast:

$$M_x = \int y\,dA \quad \gg\gg \quad \int_a^b \tfrac{1}{2}f(x)\cdot f(x)\,dx \quad \gg\gg \quad \int_a^b \tfrac{1}{2}f^2\,dx$$

- If the vertical strips are between $f_1(x)$ and $f_2(x)$, then the *midpoints* of the strips are given by the 'average' function:

$$\frac{1}{2}\Big(f_1(x) + f_2(x)\Big)$$

- The *height* of each strip is $f_2(x) - f_1(x)$, so $dA = (f_2 - f_1)\,dx$.
- Putting this together:

$$M_x = \int y\,dA \quad \gg\gg \quad \int_a^b \tfrac{1}{2}(f_1 + f_2)\cdot(f_2 - f_1)\,dx$$

$$\gg\gg \quad \int_a^b \tfrac{1}{2}\left(f_2^2 - f_1^2\right)dx$$

06 Illustration

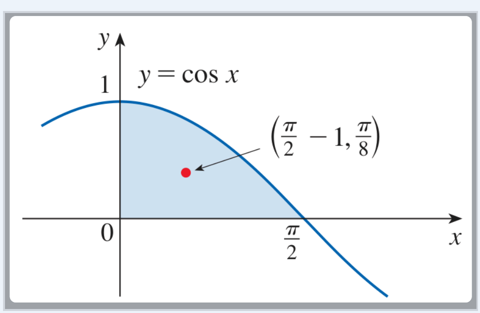
Example - Computing CoM using only vertical strips

$$M_x = \int_0^{\pi/2} \tfrac{1}{2}\rho\cos^2 x\,dx$$

$$M_y = \int_0^{\pi/2} \rho x\cos x\,dx$$

$$M = \int_0^{\pi/2} \rho\cos x\,dx = \rho$$

Find the CoM of the region:



Solution

(1) Compute the total mass M:

Area under the curve times density ρ:

$$\int_0^{\pi/2} \rho\,\cos x\,dx = \rho\sin x\Big|_0^{\pi/2} = \rho$$

(2) Compute M_y using vertical strips:

Plug $f(x) = \cos x$ into formula:

$$M_y = \int_a^b \rho\, x\, f(x)\, dx \quad \ggg \quad \int_0^{\pi/2} \rho\, x \cos x\, dx$$

Integration by parts. Set $u = x$, $v' = \cos x$ and so $u' = 1$, $v = \sin x$:

$$\int_0^{\pi/2} \rho\, x \cos x\, dx \quad \ggg \quad \rho\, x \sin x\Big|_0^{\pi/2} - \rho \int_0^{\pi/2} \sin x\, dx$$

$$\ggg \qquad \frac{\pi\rho}{2} \cdot 1 - \rho\left(-\cos\frac{\pi}{2} - -\cos 0\right) = \rho\left(\frac{\pi}{2} - 1\right)$$

---

(3) Compute M_x, also using vertical strips:

Plug $f_2(x) = f(x) = \cos x$ and $f_1(x) = 0$ into formula:

$$M_x = \int_0^{\pi/2} \rho\, \tfrac{1}{2} f_2^2\, dx \quad \ggg \quad \int_0^{\pi/2} \rho\, \tfrac{1}{2}\cos^2 x\, dx$$

Integration by 'power to frequency conversion':

$$\int_0^{\pi/2} \rho\, \tfrac{1}{2}\cos^2 x\, dx \quad \ggg \quad \frac{\rho}{4}\int_0^{\pi/2} (1 + \cos 2x)\, dx$$

$$\ggg \qquad \frac{\rho}{4} x\Big|_0^{\pi/2} + \frac{\rho \sin 2x}{8}\Big|_0^{\pi/2} = \frac{\pi\rho}{8}$$

---

(4) Compute CoM:

CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Plug in data:

$$\bar{x} = \frac{\rho(\pi/2 - 1)}{\rho} \quad \ggg \quad \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{\pi\rho/8}{\rho} \quad \ggg \quad \frac{\pi}{8}$$

$$\text{CoM} \ = \ (\bar{x}, \bar{y}) \ = \ \boxed{\left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)}$$

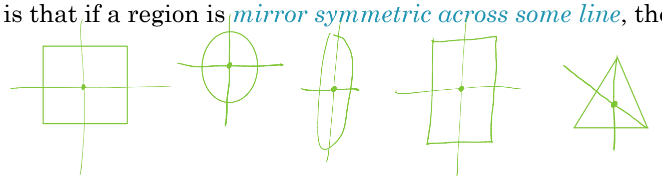
07 Theory

Two useful techniques for calculating moments and (thereby) CoMs:

- Additivity principle
- Symmetry

Additivity says that you can *add moments of parts* of a region to get the *total moment* of the region (to a given axis).

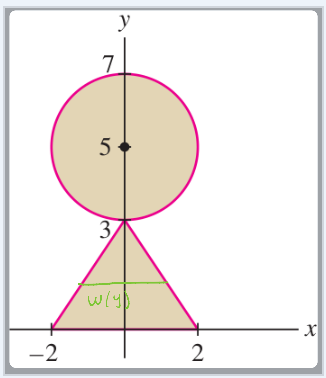
A symmetry principle is that if a region is *mirror symmetric across some line*, then the CoM must lie on that line.



08 Illustration

☰ Example - Center of mass using moments and symmetry

Find the center of mass of the two-part region:



1) $M = \rho A = \rho(4\pi + 6)$

2) $\bar{x} = 0 \ggg M_y = 0$ by symmetry

$\bar{x}^{tri} = 0 = M_y^{tri}$

$\bar{x}^{circ} = 0 = M_y^{circ}$

3) $\bar{y}^{circ} = 5 = \frac{M_x^{circ}}{M^{circ}} = \frac{?}{\rho 4\pi}$

$\ggg M_x^{circ} = 20\pi\rho$

4) $M_x^{tri} = ?$ $\quad w(y) = 4 + \frac{0-4}{3} y$

$= 4 - \frac{4}{3} y$

$dM = \rho(4 - \frac{4}{3}y)\,dy$

$M_x^{tri} = \int_0^3 y\rho(4 - \frac{4}{3}y)\,dy$

$= \rho\left(4\frac{y^2}{2} - \frac{4}{9}y^3\right)\Big|_0^3 = 4\rho\left(\frac{9}{2} - 3\right)$

$= 4\rho\frac{3}{2} \ggg 6\rho$

5) $M_x = M_x^{tri} + M_x^{circ} = 6\rho + 20\pi\rho$

$\ggg \bar{y} = \frac{6 + 20\pi}{6 + 4\pi} = \frac{3 + 10\pi}{3 + 2\pi} \approx 3.7$

Solution

(1) Symmetry: CoM on y-axis

Because the region is symmetric in the y-axis, the CoM must lie on that axis. Therefore $\bar{x} = 0$.

---

(2) Additivity of moments:

Write M_x for the total x-moment (distance measured to the x-axis from above).

Write M_x^{tri} and M_x^{circ} for the x-moments of the triangle and circle.

Additivity of moments equation:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}}$$

---

(3) Find moment of the circle M_x^{circ}:

By symmetry we know $\bar{x}^{\text{circ}} = 0$.

By symmetry we know $\bar{y}^{\text{circ}} = 5$.

Area of circle with $r = 2$ is $A = 4\pi$, so total mass is $M = 4\pi\rho$.

Centroid-from-moments equation:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M}$$

Solve the equation for M_x^{circ}:

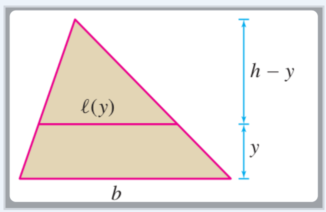
$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M} \quad \ggg \quad 5 = \frac{M_x^{\text{circ}}}{4\pi\rho}$$

$$\ggg \quad M_x^{\text{circ}} = 20\pi\rho$$

---

(4) Find moment of the triangle M_x^{tri} using integral formula:

Similar triangles:



Quick linear interpolation function:

$$\ell(y) \; = \; 0 + \frac{b-0}{h}(-y+h)$$

$$\ggg \quad \ell(y) \; = \; b - \frac{b}{h}y$$

Thus:

$$M_x^{\text{tri}} = \rho \int_0^h y\,\ell(y)\,dy \quad \ggg \quad \rho \int_0^h y\left(b - \frac{b}{h}y\right) dy$$

$$\ggg \quad \rho\left(\frac{by^2}{2} - \frac{by^3}{3h}\right)\Big|_0^h \quad \ggg \quad \frac{\rho b h^2}{6}$$

Conclude:

$$M_x^{\text{tri}} \quad \ggg \quad \frac{\rho b h^2}{6} \quad \ggg \quad \frac{\rho 4 \cdot 3^2}{6} \quad \ggg \quad 6\rho$$

---

(5) Apply additivity:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}} \quad \ggg \quad \rho(20\pi + 6)$$

---

(6) Total mass of region:

Area of circle is 4π. Area of triangle is $\frac{1}{2} \cdot 4 \cdot 3 = 6$. Thus $M = \rho A = \rho(4\pi + 6)$.

(7) Compute center of mass $\bar{y}$ from total M_x and total M:

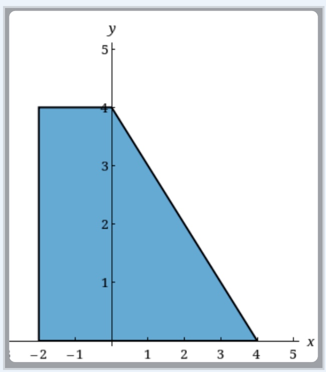
We have $M_x = \rho(20\pi + 6)$ and $M = \rho(4\pi + 6)$. Plug into formula:

$$\bar{y} = \frac{M_x}{M} \quad \ggg \quad \frac{\rho(20\pi + 6)}{\rho(4\pi + 6)} \approx 3.71$$

$$\text{CoM} \; = \; (\bar{x}, \bar{y}) = \boxed{(0, 3.71)}$$

☰ Example - Center of mass - two part region

Find the center of mass of the region which combines a rectangle and triangle (as in the figure) *by computing separate moments*. What are those separate moments? Assume the mass density is $\rho = 1$.



Solution

(1) Apply symmetry to rectangle:

By symmetry, the center of mass of the rectangle is located at $(-1, 2)$.

Thus $\bar{x}^{\text{rect}} = -1$ and $\bar{y}^{\text{rect}} = 2$.

---

(2) Find moments of the rectangle:

Total mass of rectangle $= M^{\text{rect}} = \rho \times \text{area} = 1 \cdot 8 = 8$. Thus:

$$\bar{x}^{\text{rect}} = \frac{M_y^{\text{rect}}}{M^{\text{rect}}} \qquad \ggg \qquad M_y^{\text{rect}} = -8$$

$$\bar{y}^{\text{rect}} = \frac{M_x^{\text{rect}}}{M^{\text{rect}}} \qquad \ggg \qquad M_x^{\text{rect}} = 16$$

---

(3) Find moments of the triangle:

Area of vertical slice $= \left(4 - \frac{4}{4}x\right)\,dx$. Distance from y-axis $= x$. Total M_y^{tri} integral:

$$M_y^{\text{tri}} \quad \ggg \quad \int_0^4 \rho x \left(4 - \frac{4}{4}x\right)dx$$

$$\ggg \quad \int_0^4 1\cdot(4-x)x\,dx = \frac{32}{3}$$

Total M_x^{tri} integral:

$$M_x^{\text{tri}} = \int_0^4 \rho \frac{1}{2} f(x)^2\,dx \quad \ggg \quad \int_0^4 \rho \frac{1}{2}\left(4 - \frac{4}{4}x\right)^2 dx$$

$$\ggg \quad 1\cdot\frac{1}{2}\int_0^4 (16 - 8x + x^2)\,dx \quad \ggg \quad \frac{32}{3}$$

---

(4) Add up total moments:

General formulas: $M_x = M_x^{\text{tri}} + M_x^{\text{rect}}$ and $M_y = M_y^{\text{tri}} + M_y^{\text{rect}}$

Plug in data: $M_x = \frac{32}{3} + 16 = \frac{80}{3}$ and $M_y = \frac{32}{3} - 8 = \frac{8}{3}$

---

(5) Find center of mass from moments:

Total mass of triangle $= M^{\text{tri}} = \rho \times \text{area} = 1\cdot\frac{1}{2}\cdot 4\cdot 4 = 8$.

Total combined mass $= M = M^{\text{tri}} + M^{\text{rect}} = 8 + 8 = 16$.

Apply moment relation:

$$\bar{x} = \frac{M_y}{M} \quad \ggg \quad \frac{8/3}{16} \quad \ggg \quad \frac{1}{6}$$

$$\bar{y} = \frac{M_x}{M} \quad \ggg \quad \frac{80/3}{16} \quad \ggg \quad \frac{5}{3}$$

$$\text{CoM} \;=\; (\bar{x},\bar{y}) = \boxed{\left(\frac{1}{6},\frac{5}{3}\right)}$$

Improper integrals

Videos

Videos, Math Dr. Bob:

- Improper integrals: Infinite limits
- Improper integrals: Vertical asymptote
- Improper integrals: $\int \frac{1}{x^p}\,dx$

- Improper integrals: $\int \frac{1}{x^2} e^{-1/x}\, dx$

03 Theory

Improper integrals are those for which either a *bound* or the *integrand* itself become *infinite* somewhere on the interval of integration.

Examples:

$$(a) \quad \int_1^\infty \frac{1}{x^2}\, dx, \qquad\qquad (b) \quad \int_0^2 \frac{1}{x}\, dx, \qquad\qquad (c) \quad \int_{-1}^{+1} \frac{1}{x^2}\, dx$$

(a) the upper bound is ∞
(b) the integrand goes to ∞ as $x \to 0^+$
(c) the integrand is ∞ at the point $0 \in [-1, 1]$

The limit interpretation of (a) is this:

$$\int_1^\infty \frac{1}{x^2}\, dx \quad = \quad \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\, dx$$

The limit interpretation of (b) is this:

$$\int_0^2 \frac{1}{x}\, dx \quad = \quad \lim_{R\to 0^+} \int_R^2 \frac{1}{x}\, dx$$

WRONG

① $\left.\frac{x^{-1}}{-1}\right|_{-1}^{+1} = -1\left(\frac{1}{1} - \frac{1}{-1}\right) = -2$

② $\lim\limits_{R\to\infty} \int_{-R}^{+R} \frac{1}{x^2}\, dx$

$= \lim\limits_{R\to\infty} \left. -\frac{1}{x}\right|_{-R}^{+R}$

$= \lim\limits_{R\to\infty} \left(-\frac{1}{R} - -\frac{1}{-R} = -\frac{2}{R}\right) = 0$

The limit interpretation of (c) is this:

$$\int_{-1}^{+1} \frac{1}{x^2}\, dx \quad = \quad \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_0^{+1} \frac{1}{x^2}\, dx$$

$$= \quad \lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x^2}\, dx + \lim_{R\to 0^+} \int_R^{+1} \frac{1}{x^2}\, dx$$

These limits are evaluated using the usual methods.

An improper integral is said to be **convergent** or **divergent** according to whether it may be assigned a finite value through the appropriate *limit interpretation*.

For example, (a) converges while (b) diverges.

04 Illustration

Example - Improper integral - infinite bound

Show that the improper integral $\int_2^\infty \frac{dx}{x^3}$ converges. What is its value?

Solution

(1) Replace infinity with a new symbol R:

Must write "$\lim \int \dots dx$"

Compute the integral:

$$\int_2^R \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\bigg|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

---

(2) Take limit as $R \to \infty$:

$$\lim_{R\to\infty} \frac{1}{8} - \frac{1}{2R^2} = \frac{1}{8}$$

---

(3) Improper integral definition:

$$\int_2^\infty \frac{dx}{x^3} \quad \gg\gg \quad \lim_{R\to\infty}\int_2^R \frac{dx}{x^3} \quad \gg\gg \quad \frac{1}{8}$$

Therefore that $\int_2^\infty \frac{dx}{x^3}$ converges and equals $1/8$.

☰ Improper integral - infinite integrand

Show that the improper integral $\displaystyle\int_0^9 \frac{dx}{\sqrt{x}}$ converges. What is its value?

$$\lim_{R\to 0^+} \int_R^9 \frac{dx}{\sqrt{x}}$$

Solution

(1) Replace the 0 where $\frac{1}{\sqrt{x}}$ diverges with a new symbol a:

$$R \int_a^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad \int_a^9 x^{-1/2}\,dx \; R$$

$$\gg\gg \quad 2x^{+1/2}\bigg|_a^9 \quad \gg\gg \quad 6 - 2\sqrt{a}$$

---

(2) Take limit as $a \to 0^+$:

$$\lim_{a\to 0^+} 6 - 2\sqrt{a} = 6$$

---

(3) Improper integral definition:

$$\int_{\cancel{a}\,0}^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad \lim_{a\to 0^+}\int_a^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad 6$$

Conclude that $\int_0^9 \frac{dx}{\sqrt{x}}$ converges to 6.

☰ Example - Improper integral - infinity inside the interval

Does the integral $\int_{-1}^{+1} \frac{1}{x}\, dx$ converge or diverge?

Solution

(1) WRONG APPROACH:

It is *tempting* to compute the integral *incorrectly*, like this:

$$\int_{-1}^{+1} \frac{1}{x}\, dx = \ln|x|\Big|_{-1}^{+1} = \ln|2| - \ln|-2| = 0$$

But this is wrong. There is an infinite integrand at $x = 0$. We must instead break it into parts.

---

(2) Identify discontinuity (infinity) at $x = 0$:

$$\int_{-1}^{+1} \frac{1}{x}\, dx \quad \gg\gg \quad \int_{-1}^{0} \frac{1}{x}\, dx + \int_{0}^{+1} \frac{1}{x}\, dx$$

---

(3) Improper integral definition:

$$\gg\gg \quad \lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x}\, dx + \lim_{R\to 0^+} \int_{R}^{+1} \frac{1}{x}\, dx$$

---

(3) Integrate:

$$\int_{-1}^{R} \frac{1}{x}\, dx \quad \gg\gg \quad \ln|R| - \ln|-1| \quad \gg\gg \quad \ln|R|$$

$$\int_{R}^{+1} \frac{1}{x}\, dx \quad \gg\gg \quad \ln|1| - \ln|R| \quad \gg\gg \quad -\ln R$$

---

(4) Take limits:

$$\lim_{R\to 0^-} \ln|R| = -\infty, \qquad \lim_{R\to 0^+} -\ln R = +\infty$$

Neither limit is finite. For $\int_{-1}^{+1} \frac{1}{x}\, dx$ to exist we'd need *both* of these limits to be finite. So: the original integral diverges.

05 Theory

Two tools allow us to determine convergence of a large variety of integrals. They are the **comparison test** and the **p-integral cases**.

Comparison test - integrals

The comparison test says:

- When an improper integral converges, every *smaller* integral converges.
- When an improper integral diverges, every *bigger* integral diverges.

Here, smaller and bigger are comparisons of the *integrand* at all values (accounting properly for signs), and the bounds are assumed to be the same.

For example, $\int_2^\infty \frac{dx}{x^3}$ converges, and $x^4 > x^3$ implies $\frac{1}{x^4} < \frac{1}{x^3}$ (when $x > 1$), therefore the comparison test implies that $\int_2^\infty \frac{dx}{x^4}$ converges.

p-integral cases

Assume $p > 0$ and $a > 0$. We have:

$$\begin{aligned}
p > 1: &\quad \int_a^\infty \frac{dx}{x^p} \text{ converges} \quad \text{and} \quad \int_0^a \frac{dx}{x^p} \text{ diverges} \\
p < 1: &\quad \int_a^\infty \frac{dx}{x^p} \text{ diverges} \quad \text{and} \quad \int_0^a \frac{dx}{x^p} \text{ converges} \\
p = 1: &\quad \int_a^\infty \frac{dx}{x} \text{ diverges} \quad \text{and} \quad \int_0^a \frac{dx}{x} \text{ diverges}
\end{aligned}$$

Proving the p-integral cases

It is easy to prove the convergence / divergence of each p-integral case using the limit interpretation and the power rule for integrals. (Or for $p = 1$, using $\int \frac{1}{x}\, dx = \ln x + C$.)

Additional improper integral types

$$\int_{-\infty}^{\infty} \frac{1}{x}\, dx = \lim_{R \to -\infty} \int_R^{-1} \frac{1}{x} dx + \lim_{a \to 0^-} \int_{-1}^{a} \frac{1}{x} dx + \lim_{a \to 0^+} \int_a^{1} \frac{1}{x} dx + \lim_{R \to \infty} \int_1^{\infty} \frac{1}{x} dx$$

$$\int_{-\infty}^{0} \frac{1}{x} dx + \int_0^{+\infty} \frac{1}{x} dx =$$

The improper integral $\int_{-\infty}^a f(x)\, dx$ also has a limit interpretation:

$$\int_{-\infty}^a f(x)\, dx \;=\; \lim_{R \to -\infty} \int_R^a f(x)\, dx$$

The **double improper** integral $\int_{-\infty}^\infty f(x)\, dx$ has this limit interpretation:

$$\int_{-\infty}^\infty f(x)\, dx \;=\; \lim_{R \to -\infty} \int_R^a f(x)\, dx + \lim_{R \to \infty} \int_a^R f(x)\, dx$$

Where a is any finite number. This double integral does not exist if either limit does not exist for any value of a.

⚠ **Double improper is not simultaneous!**

Watch out! This may happen:

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \neq \quad \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

This simultaneous limit might exist only because of internal cancellation in a case where the separate individual limits do not exist! We do *not* say '*convergent*' in these cases!

06 Illustration

Example - Comparison to p-integrals

Determine whether the integral converges:

(a) $\displaystyle\int_2^{\infty} \frac{x^3}{x^4-1}\,dx$

(b) $\displaystyle\int_1^{\infty} \frac{1}{x^2+x+1}\,dx$

Solution

(a)
(1) Observe large x tendency:

Consider large values. Notice the integrand tends toward $1/x$ for large x.

$$\frac{x^3}{x^4-1} \quad \longrightarrow \quad \frac{x^3}{x^4} \quad \text{for} \quad x\to\infty, \qquad \text{and} \quad \frac{x^3}{x^4} = \frac{1}{x}$$

---

(2) Try comparison to $1/x$:

$$\frac{x^3}{x^4-1} \stackrel{?}{>} \frac{1}{x}$$

Validate. Notice $x^4 - 1 > 0$ and $x > 0$ when $x \geq 2$.

$$\frac{x^3}{x^4-1} \stackrel{?}{>} \frac{1}{x}$$

$$\ggg \quad x^3 \cdot x \stackrel{?}{>} 1\cdot(x^4-1) \quad \ggg \quad x^4 \stackrel{\checkmark}{>} x^4 - 1$$

---

(3) Apply comparison test:

We know:

$$\frac{x^3}{x^4-1} > \frac{1}{x}, \qquad \int_2^{\infty} \frac{1}{x}\,dx \qquad \text{diverges}$$

We conclude:

$$\int_2^\infty \frac{x^3}{x^4-1}\,dx \qquad \text{diverges}$$

(b)
(1) Observe large x tendency:

Consider large values. Notice the integrand tends toward $1/x^2$ for large x.

$$\frac{1}{x^2+x+1} \quad \longrightarrow \quad \frac{1}{x^2} \quad \text{for} \quad x \to \infty$$

---

(2) Try comparison to $1/x^2$:

$$\frac{1}{x^2+x+1} \overset{?}{<} \frac{1}{x^2}$$

Validate. Notice $x^2+x+1>0$ and $x^2>0$ when $x \geq 1$.

$$\frac{1}{x^2+x+1} \overset{?}{<} \frac{1}{x^2}$$

$$\gg\gg \quad 1\cdot x^2 \overset{?}{<} 1\cdot(x^2+x+1) \quad \gg\gg \quad x^2 \overset{\checkmark}{<} x^2+x+1$$

---

(3) Apply comparison test:

We know:

$$\frac{1}{x^2+x+1} < \frac{1}{x^2}, \qquad \int_1^\infty \frac{1}{x^2}\,dx \qquad \text{converges}$$

We conclude:

$$\int_1^\infty \frac{1}{x^2+x+1}\,dx \qquad \text{converges}$$