Calculus II - Lecture notes - W07

Sequences

Videos

Videos, Math Dr. Bob:

- Infinite sequences: Definition; Squeeze Theorem
    - Extra: Infinite sequences: Various examples, arithmetic and geometric
    - Extra: Infinite sequences: Recursive sequences (like Fibonacci)

01 Theory

A **sequence** is a rule that defines a **term** for each natural number $n \in \mathbb{N}$:

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

So a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$.

Geometric sequence

A sequence is called **geometric** if the *ratio of consecutive terms is some constant r*, independent of n:

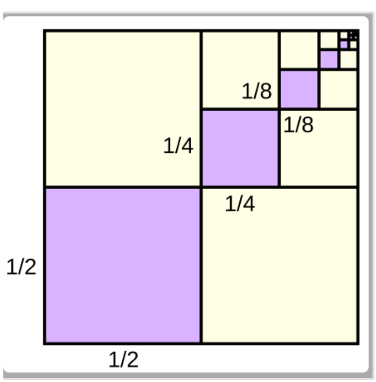
$$\frac{a_{n+1}}{a_n} = r \quad \text{for every } n$$

The defining relation of a geometric sequence is equivalent to $a_{n+1} = a_n \cdot r$.

By plugging $a_1 = a_0 \cdot r$ into $a_2 = a_1 \cdot r$, we have $a_2 = (a_0 \cdot r) \cdot r = a_0 \cdot r^2$. This plugging can be repeated n-times to get a formula for the n^{th} term:

$$a_n = a_{n-1} \cdot r = a_{n-2} \cdot r^2 = a_{n-3} \cdot r^3 = \cdots = a_1 \cdot r^{n-1} = a_0 \cdot r^n$$

Therefore $a_n = a_0 \cdot r^n$, and we have a formula for the **general term** of the sequence (the term with index n).



$a_0 = \frac{1}{4}$

$r = \frac{1}{4}$

$\text{Area} = S = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

$= \frac{a_0}{1-r} = \frac{1/4}{1-1/4} = \frac{1}{4} \cdot \frac{4}{3}$

$= \frac{1}{3}$

Starting point of a sequence

Note that sometimes the index (variable) of a sequence starts somewhere other than 0. Most common is 1 but any other starting point is allowed, even negative numbers.

Sometimes c is used instead of a_0 in the formula for the general term of a sequence, thus $a_n = cr^n$. The 'c' notation is useful when the sequence starts from $n \neq 0$.

Extra - Fibonacci sequence

The **Fibonacci** sequence goes like this:

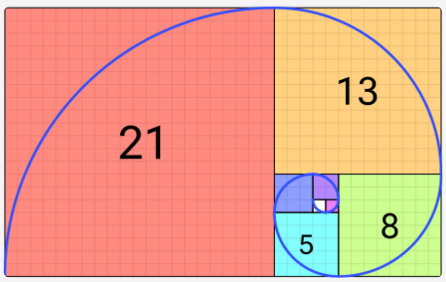
$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

The pattern is:

$$F_n = F_{n-1} + F_{n-2}$$

This formula is a **recursion relation**, which means that terms are defined using the values of prior terms.

The Fibonacci sequence is perhaps the most famous sequence of all time. It is related to the Golden Ratio and the Golden Spiral:



02 Illustration

Example: Geometric sequence - revealing the format

Find a_0 and r and a_n (written in the geometric sequence format) for the following geometric sequences:

(a) $a_n = \left(-\frac{1}{2}\right)^n$ $\quad a_0 = 1,\ r = -\frac{1}{2},\ a_n = 1 \cdot \left(-\frac{1}{2}\right)^n$

(b) $b_n = -3\left(\frac{2^{n+1}}{5^n}\right)$ $\quad b_0 = -6,\ r = \frac{2}{5},\ b_n = -6\left(\frac{2}{5}\right)^n$

(c) $c_n = e^{5+7n}$ $\quad c_0 = e^5,\ r = e^7,\ c_n = e^5(e^7)^n$

Solution $\quad r = \frac{c_{n+1}}{c_n} = \frac{e^{5+7(n+1)}}{e^{5+7n}} = \frac{e^{5+7n+7}}{e^{5+7n}} = e^7$ $\quad$ Must become $a_0 (r)^n$

(a)
Plug in $n = 0$ to obtain $a_0 = 1$. Notice that $a_{n+1}/a_n = -1/2$ and so therefore $r = -1/2$. Then the 'general term' is $a_n = a_0 \cdot r^n = 1 \cdot (-1/2)^n$.

(b)
Rewrite the fraction:

$$\frac{2^{n+1}}{5^n} \quad \ggg \quad 2 \cdot \left(\frac{2}{5}\right)^n$$

Plug that in and observe $b_n = -6 \cdot (2/5)^n$. From this format we can *read off* $b_0 = -6$ and $r = 2/5$.

(c)
Rewrite:

$$c_n \quad \gg\gg \quad e^5 \cdot e^{7n} \quad \gg\gg \quad e^5 \cdot \left(e^7\right)^n$$

From this format we can *read off* $c_0 = e^5$ and $r = e^7$.

Series

Videos

Videos, Math Dr. Bob:

- [Infinite series](): Definitions, basic examples
- [Geometric series and SDT](): Geometric series, Simple Divergence Test (aka "Limit Test")
- [Infinite series](): Various examples
  - Extra: [Infinite series convergence](): Telescoping series

03 Theory

A **series** is an infinite sum that is created by successive additions without end. The terms are not added up "all at once" but rather they are added up "as n increases" or "as $n \to \infty$."

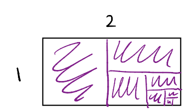
$$a_0 + a_1 + a_2 + a_3 + \dots \qquad = \quad \sum_{n=0}^{\infty} a_n$$

Three of the most famous series are the Leibniz series and the geometric series:

$$\text{Leibniz series:} \qquad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \qquad = \quad \frac{\pi}{4}$$

$$\text{Geometric series:} \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n + \cdots \qquad = \quad 2$$

area = 2



Partial sum sequence of a series

The **partial sum sequence** of a series is the *sequence* whose terms are the sums up to the given index:

$$S_N = a_0 + a_1 + \cdots + a_N \qquad = \quad \sum_{n=0}^{N} a_n$$

$S_7 - S_5 = a_6 + a_7$

$S_7 - S_5$

$\underbrace{a_0 + a_1 + a_2 + a_3 + a_4 + a_5}_{S_5} + a_6 + a_7 = S_7$

These S_N terms themselves form a sequence:

$$S_0,\ S_1,\ S_2,\ S_3,\ \dots$$

$S_N - S_{N-1} = a_N$

04 Illustration

Example: Geometric series - total sum and partial sums

The geometric series *total sum S* can be calculated using a *"shift technique"* as follows:
(1) Compare S and rS:

$$\begin{array}{rrcl} & S & = & a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots \\ \overset{\times r}{\gg\!\gg} & rS & = & a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \cdots \end{array}$$

(2) Subtract second line from first line, many cancellations:

$$\begin{array}{rcl} S & = & a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots \\ -\Big(rS & = & a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \cdots \Big) \\ \hline S - rS & = & a_0 \end{array}$$

(3) Solve to find S:

$$S = \frac{a_0}{1-r}$$

⚠ **Note: this calculation *assumes* that S exists, i.e. that the series *converges*.**

The geometric series *partial sums* can be calculated similarly, as follows:

(1) Compare S and rS:

$$\begin{array}{rrcl} & S_N & = & a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N \\ \overset{\times r}{\gg\!\gg} & rS_N & = & a_0 r + a_0 r^2 + \cdots + a_0 r^N + a_0 r^{N+1} \end{array}$$

(2) Subtract second line from first line, many cancellations:

$$\begin{array}{rcl} S_N & = & a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^N \\ -\Big(rS_N & = & a_0 r + a_0 r^2 + \cdots + a_0 r^N + a_0 r^{N+1} \Big) \\ \hline S_N - rS_N & = & a_0 - a_0 r^{N+1} \end{array}$$

(3) Solve to find S_N:

$$S_N = a_0 \frac{1 - r^{N+1}}{1-r}$$

$$= \frac{a_0}{1-r} - \frac{a_0}{1-r} r^{N+1} = S - S r^{N+1}$$

(4) The last formula is revealing in its own way. Here is what it means in terms of terms:

$$a_0 + a_0 r + \cdots + a_0 r^N = $$

$$a_0 + a_0 r + a_0 r^2 + \cdots$$

$$- \left(a_0 r^{N+1} + a_0 r^{N+2} + \cdots \right)$$

$$\begin{array}{rl} 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots & = S \\ -\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right. & \left. = \frac{1}{2} S \right) \\ \hline 1 & = S - \frac{1}{2} S \\ & = \frac{1}{2} S \end{array}$$

So $S = 2$

Convergence

Videos

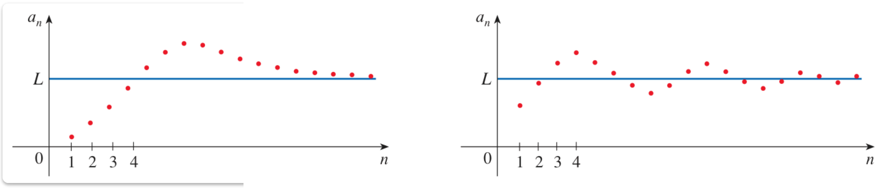
Videos, Math Dr. Bob:

- [Infinite sequences convergence]: Squeeze; Monotone Bounded

- [Infinite sequences convergence](): Examples sequences: convergent, monotonic, bounded

05 Theory

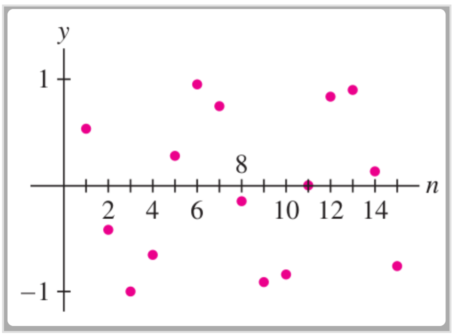
A sequence has a **limit** if its terms tend toward a specific number, or toward $\pm\infty$.



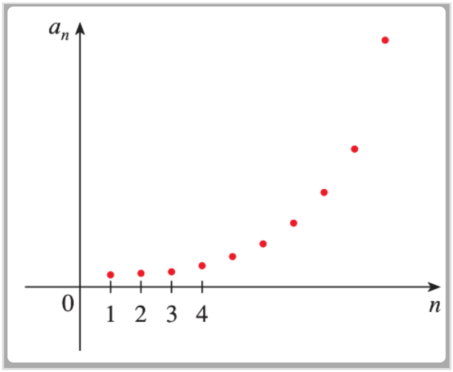
When this happens we can write "$\lim_{n\to\infty} a_n = L$" with some number $L \in \mathbb{R}$ or $L = \pm\infty$.
We can also write "$a_n \to L$ as $n \to \infty$".

The sequence is said to **converge** if it has a *finite limit* $L \in \mathbb{R}$.

Some sequences don't have a limit at all, like $a_n = \cos n$:



Or $a_n = e^n$:



These sequences **diverge**.
In the second case, there is a limit $L = \infty$, so we say it **diverges to** $+\infty$.

A sequence may *have a limit* of $\pm\infty$ but is still said to *diverge*.

Extra - Convergence definition

The precise meaning of convergence is this. We have $a_n \to L$ as $n \to \infty$ if, given any proposed error $\varepsilon > 0$, it is possible to find N such that for all $n > N$ we have $|a_n - L| < \varepsilon$.

When $L = \infty$, convergence means that given any $B > 0$, we can find N such that for all $n > N$ we have $a_n > B$.

Similarly for $L = -\infty$.

If the general term a_n is a continuous function of n, we can replace n with the continuous variable x and compute the continuous limit instead:

$$\lim_{n\to\infty} a_n \quad = \quad \lim_{x\to\infty} a_x$$

If a_x would be a *differentiable* function, and we discover an *indeterminate form*, then we can apply L'Hopital's Rule to find the limit value. For example, if the indeterminate form is $0 \cdot \infty$, we can convert it to $\frac{\infty}{1/0} = \frac{\infty}{\infty}$ and apply L'Hopital.

06 Illustration

Example - L'Hopital's Rule for sequence limits

(a) What is the limit of $a_n = \dfrac{\ln n}{n}$?

(b) What is the limit of $b_n = \dfrac{(\ln n)^2}{n}$?

(c) What is the limit of $c_n = n\left(\sqrt{n^2+1} - \sqrt{n}\right)$?

Solution

(a)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\frac{d}{dx}}{\gg\gg} \quad \lim_{x\to\infty} \frac{1/x}{1} = 0$$

(b)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \quad \overset{\frac{d}{dx}}{\gg\gg} \quad \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} \quad \overset{\text{(by } a_n \text{ result)}}{=} \quad 0$$

(c)

(1) Identify form $\infty \cdot 0$ and rewrite as $\frac{\infty}{\infty}$:

$$n\left(\sqrt{n^2+1} - \sqrt{n}\right) \quad \gg\gg \quad \frac{\sqrt{n^2+1} - \sqrt{n}}{1/n}$$

(2) Change from n to x and apply L'Hopital:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt{x}}{1/x} \quad \gg\gg \quad \frac{\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) - \frac{1}{2}x^{-1/2}}{-1/x^2}$$

(3) Simplify:

$$\gg\gg \quad \frac{-2x^3}{\sqrt{x^2+1}} + x^{3/2} = \frac{-2x^3 + x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

(4) Consider the limit:

$$\frac{-2x^3 + x^{3/2}\sqrt{x^2+1}}{\sqrt{x^2+1}} \overset{x\to\infty}{\longrightarrow} \frac{-2x^3 + x^{3/2}x}{x} \longrightarrow \frac{-2x^3}{x} \longrightarrow -\infty$$

Example - Squeeze theorem

Use the squeeze theorem to show that $\frac{4^n}{n!} \to 0$ as $n \to \infty$.

Solution

(1) We will squeeze the given general term above 0 and below a sequence b_n that we must devise:

$$0 \le \frac{4^n}{n!} \le b_n$$

(2) We need b_n to satisfy $b_n \to 0$ and $\frac{4^n}{n!} \le b_n$. Let us study $\frac{4^n}{n!}$.

$$\frac{4^n}{n!} = \frac{4\cdot 4\cdot\ \cdots\ \cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}{n(n-1)\cdots 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

(3) Now for the trick. Collect factors in the middle bunch:

$$\frac{4^n}{n!} = \frac{4}{n}\left(\frac{4}{n-1}\cdot\frac{4}{n-2}\cdot\ \cdots\ \cdot\frac{4}{7}\cdot\frac{4}{6}\cdot\frac{4}{5}\right)\frac{4\cdot 4\cdot 4\cdot 4}{4\cdot 3\cdot 2\cdot 1}$$

(4) Each factor in the middle bunch is < 1 so the entire middle bunch is < 1. Therefore:

$$\frac{4^n}{n!} < \frac{4}{n}\cdot\frac{4^4}{4!} = \frac{1024}{24n}$$

Now we can easily see that $1024/24n \to 0$ as $n \to \infty$, so we set $b_n = 1024/24n$ and we are done.