Calculus II - Lecture notes - W08

Convergence

01 Theory

Monotone sequences

A sequence is called **monotone increasing** if $a_{n+1} \geq a_n$ for every n.

A sequence is called **monotone decreasing** if $a_{n+1} \leq a_n$ for every n.

In this context, 'monotone' just means it preserves the increasing or decreasing modality for *all* terms.

Monotonicity Theorem

If a sequence is monotone increasing, and **bounded above** by B, then it *must converge* to some limit L, and $L \leq B$.

If a sequence is monotone decreasing, and **bounded below** by B, then it *must converge* to some limit L, and $L \geq B$.

Terminology:

- *Bounded above by B* means that $a_n \leq B$ for every n
- *Bounded below by B* means that $B \leq a_n$ for every n

Notice!

The Monotonicity Theorem says that a limit L *exists*, but it does not *provide* the limit value.

02 Illustration

Example - Monotonicity Theorem

Show that $a_n = \sqrt{n+1} - \sqrt{n}$ converges.

Solution

(1) Observe that $a_n > 0$ for all n.

Because $n+1 > n$, we know $\sqrt{n+1} > \sqrt{n}$.

Therefore $\sqrt{n+1} - \sqrt{n} > 0$

---

(2) Change n to x and show a_x is decreasing.

New formula: $a_x = \sqrt{x+1} - \sqrt{x}$ considered as a *differentiable* function.

⚠ **Take derivative to show decreasing.**

Derivative of a_x:

$$\frac{d}{dx}a_x = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

---

(3) Simplify:

$$\gg\gg \qquad \frac{2\left(\sqrt{x}-\sqrt{x+1}\right)}{4\sqrt{x}\sqrt{x+1}}$$

Denominator is >0. Numerator is <0. So $\frac{d}{dx}a_x < 0$ and a_x is monotone decreasing.

Therefore a_n is monotone decreasing as $n \to \infty$.

03 Theory

Series convergence

We say that a *series* converges when its *partial sum sequence converges*:

$$\text{“}\sum_{n=0}^{\infty} a_n \quad \text{converges”} \qquad \text{MEANS:} \qquad \text{“}S_N \quad \text{converges as } N \to \infty\text{”}$$

Let us apply this to the geometric series. Recall our formula for the partial sums:

$$S_N = a_0 \frac{1-r^{N+1}}{1-r}$$

Rewrite this formula:

$$\gg\gg \qquad S_N = \frac{a_0}{1-r} - \frac{a_0}{1-r}r^{N+1}$$

Now take the limit as $N \to \infty$:

$$\lim_{N\to\infty} S_N \quad = \quad \text{“}\,\frac{a_0}{1-r} - \frac{a_0}{1-r}r^{\infty+1}\,\text{”} \quad = \quad \frac{a_0}{1-r}$$

⚠ **So we see that S_N converges exactly when $|r|<1$. It converges to $\frac{a_0}{1-r}$.**

(If $|r|=1$ then the denominator is 0, and if $|r|>1$ then the factor $r^{\infty+1}$ does not converge.)

Furthermore, we have the limit value:

$$\sum_{n=0}^{\infty} a_n \quad = \quad \lim_{N\to\infty} S_N = \frac{a_0}{1-r} = S$$

This result confirms the formula we derived for the total S for a geometric series. This time we did not start by assuming S exists, on the contrary we *proved* that S exists. (Provided that $|r|<1$.)

Extra - Aspects of S and S_N from the geometric series

Notice that we always have the rule:

$$S_N = S - r^{N+1}S$$

$$S_N = \frac{a_0}{1-r} - \frac{a_0}{1-r}r^{N+1}$$

This rule can be viewed as coming from partitioning the full series into a finite part S_N and the remaining infinite part:

$$S = \underbrace{a_0 + a_0r + \cdots + a_0r^N}_{S_N} + \underbrace{a_0r^{N+1} + a_0r^{N+2} + \ldots}_{S-S_N}$$

We can remove a factor r^{N+1} from the infinite part:

$$S - S_N = r^{N+1}\left(a_0 + a_0r + a_0r^2\ \ldots\right)$$

The parenthetical expression is equal to S, so we have the formula $S_N = S - r^{N+1}S$ given above.

Simple divergence test

Videos

Videos, Math Dr. Bob

- Geometric series and SDT, again: Geometric series, Simple Divergence Test (aka “Limit Test”)
- Integral test: Basics
- Integral test: p-series
  - Extra: Integral test: Further examples
  - Extra: Integral test: Estimations

04 Theory

Simple Divergence Test (SDT)

Applicability: *Any* series.

Test Statement:

$$\lim_{n\to\infty} a_n \neq 0 \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

⚠ **The *converse is not valid*. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.**

05 Illustration

Example - Simple Divergence Test: examples

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

- This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

Consider: $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n+1}$

- This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.
- We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \ldots$," but that is not a limit value.

Positive series

Videos

- Direct Comparison Test: Theory and basic examples
- Direct Comparison Test: Series $\frac{1}{\ln n}$
- Limit Comparison Test: Theory and basic examples
- Limit Comparison Test: Further examples

06 Theory

Positive series

A series is called **positive** when its individual terms are positive, i.e. $a_n > 0$ for all n.

The partial sum sequence S_N is *monotone increasing* for a positive series.

By the monotonicity test for convergence of sequences, S_N therefore converges whenever it is *bounded above*. If S_N is not bounded above, then $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

Another test, called the **integral test**, studies the terms of a series as if they represent rectangles with upper corner pinned to the graph of a continuous function.

To apply the test, we must convert the integer index variable n in a_n into a continuous variable x. This is easy when we have a formula for a_n (provided it doesn't contain factorials or other elements dependent on integrality).

Integral Test (IT)

Applicability:

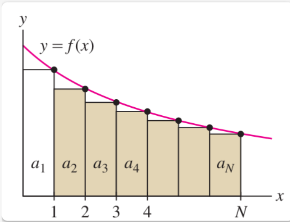
- (i) $f(x) > 0$
- (ii) $f(x)$ is continuous
- (iii) $f(x)$ is *monotone decreasing*

Test Statement:

$$\sum_{n=1}^{\infty} a_n \quad \frac{\text{converges}}{\text{diverges}} \qquad \iff \qquad \int_1^{\infty} f(x)\,dx \quad \frac{\text{converges}}{\text{diverges}}$$

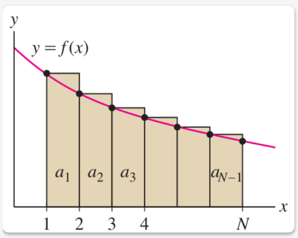
Extra - Integral test: explanation

To show that *integral convergence implies series convergence*, consider the diagram:



This shows that $\sum_{n=2}^{N} a_n \leq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ converges, then $\int_1^N f(x)\,dx$ is bounded (independent of N) and so $\sum_{n=2}^{N} a_n$ is bounded by that inequality. But $\sum_{n=2}^{N} a_n = S_N - a_1$; so by adding a_1 to the bound, we see that S_N itself is bounded, which implies that $\sum_{n=1}^{\infty} a_n$ converges.

To show that *integral divergence implies series divergence*, consider a similar diagram:



This shows that $\sum_{n=1}^{N-1} a_n \geq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^N f(x)\,dx$ goes to $+\infty$ as $N \to \infty$, and so $\sum_{n=1}^{N-1} a_n$ goes to $+\infty$ as well. So $\sum_{n=1}^{\infty} a_n$ diverges.

Notice: the picture shows $f(x)$ entirely above (or below) the rectangles.

This depends upon $f(x)$ being *monotone decreasing*, as well as $f(x) > 0$.
This explains the applicability conditions.

Next we use the integral test to evaluate the family of ***p*-series**, and later we can use p-series in comparison tests without repeating the work of the integral test.

p-series

A ***p*-series** is a series of this form: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

Convergence properties:

$$p > 1 : \text{series converges} \qquad\qquad p \leq 1 : \text{series diverges}$$

Extra - Proof of p-series convergence

(1) To verify the convergence properties of p-series, apply the integral test:

Applicability: verify it's continuous, positive, decreasing.

Convert n to x to obtain the function $f(x) = \frac{1}{x^p}$.

Indeed $\frac{1}{x^p}$ is continuous and positive and decreasing as x increases.

(2) Apply the integral test.

Integrate, assuming $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \ggg \quad \lim_{R\to\infty} \left.\frac{x^{p-1}}{p-1}\right|_1^R$$

$$\ggg \quad \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

When $p > 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = 0$

When $p < 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = \infty$

When $p = 1$, integrate a second time:

$$\int_1^\infty \frac{1}{x}\,dx \quad \ggg \quad \lim_{R\to\infty} \left.\ln x\right|_1^R$$

$$\ggg \quad \lim_{R\to\infty} \ln R - \ln 1 \quad \ggg \quad \infty$$

---

(3) Conclude: the integral converges when $p > 1$ and diverges when $p \leq 1$.

Supplement: we could instead immediately refer to the convergence results for *p-integrals* instead of reproving them here.

07 Illustration

Example - p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^\infty \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p \leq 1$

Example - Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ converge?

Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n\ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

(1) The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.

Check applicability.

Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty)$.

---

(2) Apply the integral test to $f(x)$.

Integrate $f(x)$:

$$\int_2^\infty \frac{1}{x \ln x}\, dx \quad \ggg \quad \int_{u=\ln 2}^\infty \frac{1}{u}\, du$$

$$\ggg \quad \lim_{R\to\infty} \ln u \Big|_{\ln 1}^{R} \quad \ggg \quad \infty$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n \ln n}$ *diverges*.

---

(3) Apply the integral test to $g(x)$.

Integrate $g(x)$:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx \quad \ggg \quad \int_{u=\ln 2}^\infty \frac{1}{u^2}\, du$$

$$\ggg \quad \lim_{R\to\infty} -u^{-1} \Big|_{\ln 2}^{R} \quad \ggg \quad \frac{1}{\ln 2}$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ *converges*.