Calculus II - Lecture notes - W09

Positive series

01 Theory

Direct Comparison Test (DCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose $a_n \leq b_n$ for large enough n.
(Meaning: for $n \geq N$ with some given N.) Then:

- Smaller pushes up bigger:

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} b_n \quad \text{diverges}$$

- Bigger controls smaller:

$$\sum_{n=1}^{\infty} b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{converges}$$

02 Illustration

Example - Direct comparison test: rational functions

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ *converges* by the DCT.

Choose: $a_n = \frac{1}{\sqrt{n}\,3^n}$ and $b_n = \frac{1}{3^n}$

Check: $0 < \frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$

Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ *converges* by the DCT.

Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.

Check: $\cancel{0 <} \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$

Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

Large n,
Looks like
$\frac{n}{n^3}$ i.e. $\frac{1}{n^2}$

(c) The series $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ *converges* by the DCT.

$\sum_{n=2}^{\infty} \frac{n}{n^3-1}$

① reindex

$= \sum_{n+1=2}^{\infty} \frac{n+1}{(n+1)^3-1}$

$= \sum_{n=1}^{\infty} \frac{n+1}{n^3+3n^2+3n+1-1}$

$= \sum_{n=1}^{\infty} \frac{n+1}{n^3+3n^2+3n}$

$\leq \sum_{n=1}^{\infty} \frac{n+1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. conv.

$\Rightarrow$ conv.

②

$\frac{n}{n^3-1} \leq \frac{10}{n^2}$ (for $n \geq 2$)

cross $\times$

$n^3 \leq 10n^3 - 10$

$10 \leq 9n^3$ ✓ $n \geq 2$

know $\sum_{n=2}^{\infty} \frac{10}{n^2}$ conv.

Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$

Check: $\cancel{0 \leq}\ \frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)

Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

(d) The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ *diverges* by the DCT.

Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$

Check: $0 \leq \frac{1}{n} \leq \frac{1}{n-1}$

Observe: $\sum \frac{1}{n}$ is a divergent p-series

03 Theory

Some series can be compared using the DCT after applying certain manipulations and tricks.

For example, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. We suspect convergence because $a_n \approx \frac{1}{n^2}$ for *large* n. But unfortunately, $a_n > \frac{1}{n^2}$ always, so we cannot apply the DCT.

We could make some *ad hoc* arguments that do use the DCT, eventually:

- Trick Method 1:
  - Observe that for $n > 1$ we have $\frac{1}{n^2-1} \leq \frac{10}{n^2}$. (Check it!)
  - But $\sum \frac{10}{n^2}$ converges, indeed its value is $10 \cdot \sum \frac{1}{n^2}$, which is $\frac{10\pi^2}{6}$.
  - So the series $\sum \frac{1}{n^2-1}$ converges.
- Trick Method 2:
  - Observe that we can change the letter n to $n+1$ by starting the new n at $n = 1$.
  - Then we have:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

  - This last series has terms smaller than $\frac{1}{n^2}$ so the DCT with $b_n = \frac{1}{n^2}$ (a convergent p-series) shows that the original series converges too.

These convoluted arguments suggest that a more general version of Comparison is possible.

Indeed, it is sufficient to compare the *limiting behavior* of two series. The limit of *ratios* (limit of 'comparison') links up the convergence / divergence of $\sum a_n$ and $\sum b_n$.

Limit Comparison Test (LCT) - "Limiting Ratio Test"

$L_n = \frac{a_n}{b_n} \rightsquigarrow a_n = L_n \cdot b_n$

$L_n \longrightarrow L$

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then:

Large n, have

$a_n \approx L \cdot b_n$

if $L = 0$, b_n conv.
$\Rightarrow a_n$ conv.

if $L = \infty$, b_n div.
$\Rightarrow a_n$ div.

- If $0 < L < \infty$:

$$\sum a_n \quad \text{converges} \qquad \Longleftrightarrow \qquad \sum b_n \quad \text{converges}$$

a_n/b_n

$\|$

$L_n \longrightarrow L$

$a_n = L_n b_n$

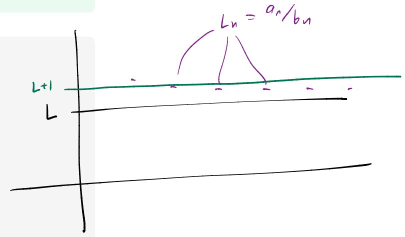
If $L = 0$ or $L = \infty$, we can still draw an inference, but in only one direction:

- If $L = 0$:

$$\sum b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{converges}$$

- If $L = \infty$:

$$\sum b_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{diverges}$$



Extra - Limit Comparison Test: Theory

Suppose $a_n/b_n \to L$ and $0 < L < \infty$. Then for n sufficiently large, we know $a_n/b_n < L + 1$.

$a_n < (L+1)\, b_n$

Doing some algebra, we get $a_n < (L+1)b_n$ for n large.

If $\sum b_n$ converges, then $\sum (L+1)b_n$ also converges (constant multiple), and then the DCT implies that $\sum a_n$ converges.

Conversely: we also know that $b_n/a_n \to 1/L$, so $b_n < (1/L + 1)a_n$ for all n sufficiently large. Thus if $\sum a_n$ converges, $\sum (1/L + 1)a_n$ also converges, and by the DCT again $\sum b_n$ converges too.

The cases with $L = 0$ or $L = \infty$ are handled similarly.

04 Illustration

Example - Limit Comparison Test examples

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ *converges* by the LCT.

Choose: $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$.

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{2^n}{2^n - 1} \quad \gg\gg \quad 1 \;=:\; L$$

Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

---

Large n looks like $\longrightarrow \dfrac{2n^2}{\sqrt{n^5}} = \dfrac{2}{n^{1/2}}$

Precisely:

$$\frac{\left(\frac{2n^2+3n}{\sqrt{5+n^5}}\right)}{\left(\frac{2}{n^{1/2}}\right)} = \frac{(n^2 + \frac{3}{2}n)\, n^{1/2}}{\sqrt{5+n^5}} = \frac{n^{5/2} + \frac{3}{2} n^{3/2}}{\sqrt{5+n^5}} \xrightarrow{n\to\infty} \frac{n^{5/2}}{n^{5/2}} \longrightarrow 1 = L$$

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ *diverges* by the LCT.

Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = n^{-1/2}$ $= \dfrac{1}{n^{1/2}}$

Compare in the limit:

$$\lim_{n\to\infty}\frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty}\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \xrightarrow{n\to\infty} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2 \;=:\; L$$

Observe: $\sum n^{-1/2}$ is a divergent p-series

(c) The series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{n^4-n-1}$ *converges* by the LCT.

Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = n^{-2}$ $= \frac{1}{n^2}$

Compare in the limit:

$$\lim_{n\to\infty}\frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty}\frac{n^4}{n^4-n-1} \quad \ggg \quad 1 \;=:\; L$$

Observe: $\sum_{n=2}^{\infty} n^{-2}$ is a converging p-series

Alternating series

Videos

Videos, Math Dr. Bob:

- Alternating Series Test: Theory and basic examples
- Alternating Series Test: Remainder estimates
- Alternating Series Test: Further remainder estimates

05 Theory

Consider these series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad \infty$$

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\cdots \quad = \quad -\infty$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots \quad = \quad \ln 2$$

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad ?$$

The absolute values of terms are the same between these series, only the signs of terms change.

The first is a **positive series** because there are no negative terms.

The second series is the negation of a positive series – the study of such series is equivalent to that of positive series, just add a negative sign everywhere. These signs can be factored out of the series. (For example $\sum -\frac{1}{n} = -\sum \frac{1}{n}$.)

every (alt.) : $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ $(a_n > 0)$

The third series is an **alternating series** because the signs alternate in a strict pattern, every other sign being negative.

The fourth series is *not* alternating, nor is it positive, nor negative: it has a mysterious or unknown pattern of signs.

A series with any negative signs present, call it $\sum_{n=1}^{\infty} a_n$, **converges absolutely** when the positive series of absolute values of terms, namely $\sum_{n=1}^{\infty} |a_n|$, converges.

> **THEOREM: Absolute implies ordinary**
>
> If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it also converges as it stands.

A series might converge due to the presence of negative terms and yet *not* converge absolutely:

A series $\sum_{n=1}^{\infty} a_n$ is said to be **converge conditionally** when the series converges as it stands, but the series produced by inserting absolute values, namely $\sum_{n=1}^{\infty} |a_n|$, diverges.

The alternating harmonic series above, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \;=\; \ln 2$, is therefore conditionally convergent. Let us see why it converges. We can group the terms to create new sequences of *pairs*, each pair being a positive term. This can be done in two ways. The first creates an increasing sequence, the second a decreasing sequence:

$$\text{increasing from 0:} \qquad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots$$

$$\text{decreasing from 1:} \qquad 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots$$

Suppose S_N gives the sequence of partial sums of the original series. Then S_{2N} gives the first sequence of pairs, namely $S_2, S_4, S_6, \ldots$. And S_{2N-1} gives the second sequence of pairs, namely $S_1, S_3, S_5, \ldots$.

The second sequence shows that S_N is bounded above by 1, so S_{2N} is monotone increasing and bounded above, so it converges. Similarly S_{2N-1} is monotone decreasing and bounded below, so it converges too, and of course they must converge to the same thing.

The fact that the terms were *decreasing in magnitude* was an essential ingredient of the argument for convergence. This fact ensured that the parenthetical pairs were positive numbers.

> **Alternating Series Test (AST) - "Leibniz Test"**
>
> **Applicability:** Alternating series only: $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $a_n > 0$
>
> **Test Statement:**
> If:
>
> 1. a_n are *decreasing*, so $a_1 > a_2 > a_3 > a_4 > \cdots > 0$
> 2. $a_n \to 0$ as $n \to \infty$ (i.e. it passes the SDT)
>
> Then:
>
> $$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{converges}$$
>
> Furthermore, partial sum *errors* are bounded by "subsequent terms":

$$|S - S_N| \leq a_{N+1}$$

"next term bound"

Extra - Alternating Series Test: Theory

Just as for the alternating harmonic series, we can form *positive* paired-up series because the terms are decreasing:

$$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots$$

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots$$

The first sequence S_{2N} is monotone increasing from 0, and the second S_{2N-1} is decreasing from a_1. The first is therefore also bounded above by a_1. So it converges. Similarly, the second converges. Their difference at any point is $S_{2N} - S_{2N-1}$ which is equal to $-a_{2N}$, and this goes to zero. So the two sequences must converge to the same thing.

By considering these paired-up sequences and the effect of adding each new term one after the other, we obtain the following order relations:

$$0 < S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1 = a_1$$

Thus, for *any even* $2N$ and *any odd* $2M - 1$:

$$S_{2N} < S < S_{2M-1}$$

Now set $M = N$ and subtract S_{2N-1} from both sides:

$$S_{2N} - S_{2N-1} < S - S_{2N-1} < 0$$

$$\gg\gg \quad -a_{2N} < S - S_{2N-1} < 0$$

Now set $M = N + 1$ and subtract S_{2N} from both sides:

$$0 < S - S_{2N} < S_{2N+1} - S_{2N}$$

$$\gg\gg \quad 0 < S - S_{2N} < a_{2N+1}$$

This covers both even cases ($n = 2N$) and odd cases ($n = 2N - 1$). In either case, we have:

$$|S - S_n| < a_{n+1}$$

06 Illustration

Example - Alternating Series Test: Basic illustration

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.

Therefore the first series converges *conditionally*.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converges by the AST.

Notice the funny notation: $\cos n\pi = (-1)^n$.

This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Example - Approximating π

The Taylor series for $\tan^{-1} x$ is given by:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate π with an error less than 0.001.

Solution

(1) The main idea is to use $\tan \frac{\pi}{4} = 1$ and thus $\tan^{-1} 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

---

(2) Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. Therefore, calculate n such that $a_{n+1} < 0.001$, and then we will know:

$$|E_n| < a_{n+1} < 0.001$$

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(3) The general term is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Now solve:

$$a_{n+1} = \frac{4}{2n+1} < 0.001$$

$$\ggg \qquad \frac{4}{0.001} < 2n + 1$$

$$\ggg \qquad 3999 < 2n$$

$$\ggg \qquad 2000 \leq n$$

We conclude that at least 2000 terms are necessary to be confident (by the error formula) that the approximation of π is accurate to within 0.001.

Ratio test and Root test

Videos

Videos, Math Dr. Bob

- Ratio test: Basics
- Ratio test: Ratio test + DCT
- Root test: Basics
- Root test: for $\sum(1-1/n^2)^{n^3}$

07 Theory

Ratio Test (RaT)

Applicability: Any series with nonzero terms.

Test Statement:

Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$ as $n \to \infty$.

Then:

$$\begin{aligned} L < 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely} \\ L > 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{diverges} \\ L = 1 \text{ or DNE}: &\quad \text{test inconclusive} \end{aligned}$$

Extra - Ratio test: explanation

To understand the ratio test, consider this series:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \quad = \quad 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots$$

$\frac{2 \cdot 2^2}{3 \cdot 2!} = \left(\frac{2}{3}\right) a_2$

$\frac{2 \cdot 2^3}{4 \cdot 3!} = \left(\frac{2}{4}\right) a_3$

$= \left(\frac{2}{4}\right)\left(\frac{2}{3}\right) a_2$

$\vdots$

ratio is

$R_n = \frac{2}{n} \longrightarrow 0$

- The term $\frac{2^3}{3!}$ is given by multiplying the prior term by $\frac{2}{3}$.
- The term $\frac{2^4}{4!}$ is given by multiplying the prior term by $\frac{2}{4}$.
- The term a_n is created by multiplying the prior term by $\frac{2}{n}$.

When $n > 3$, the multiplication factor giving the next term is necessarily less than $\frac{2}{3}$. Therefore, when $n > 3$, the terms shrink *faster than those of a geometric series* having $r = \frac{2}{3}$. Therefore this series converges.

Similarly, consider this series:

$$\sum_{n=0}^{\infty} \frac{10^n}{n!} \quad = \quad 1 + \frac{10}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots$$

Write $R_n = \frac{a_n}{a_{n-1}}$ for the ratio from the prior term a_{n-1} to the current term a_n. For this series, $R_n = \frac{10}{n}$.

This ratio falls below $\frac{10}{11}$ when $n > 11$, after which the terms necessarily shrink faster than those of a geometric series with $r = \frac{10}{11}$. Therefore this series converges.

The main point of the discussion can be stated like this:

$$R_n \to L < 1 \quad \text{as} \quad n \to \infty$$

Whenever this is the case, then *eventually* the ratios are bounded below some $r < 1$, and the series terms are smaller than those of a converging geometric series.

Extra - Ratio test: proof

Let us write $R_n = \left|\frac{a_{n+1}}{a_n}\right|$ for the ratio to the next term from term n.

Suppose that $R_n \to L$ as $n \to \infty$, and that $L < 1$. This means: eventually the ratio of terms is close to L; so eventually it is less than 1.

More specifically, let us define $r = \frac{L+1}{2}$. This is the point halfway between L and 1. Since $R_n \to L$, we know that eventually $R_n < r$.

Any geometric series with ratio r converges. Set $c = a_N$ for N big enough that $R_N < r$. Then the terms of our series satisfy $|a_{N+n}| \leq cr^n$, and the series starting from a_N is absolutely convergent by comparison to this geometric series.

(Note that the terms $a_1, \ldots, a_{N-1}$ do not affect convergence.)

08 Illustration

Example - Ratio test

(a) Observe that $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n+1}$ and thus $R_n \to 0 = L < 1$. Therefore the RaT implies that this series converges.

Simplify the ratio:

$$\frac{\frac{10^{n+1}}{(n+1)!}}{\cancel{\frac{n!}{10^n}}\ \frac{10^n}{n!}} \quad \gg\gg \quad \overset{\frac{10^{n+1}}{(n+1)!}}{\cancel{\frac{(n+1)!}{10^{n+1}}}} \cdot \frac{n!}{10^n}$$

$$\gg\gg \quad \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \quad \gg\gg \quad \frac{10}{n+1} \xrightarrow{n\to\infty} 0$$

Notice this technique! We *frequently* use these rules:

$$10^{n+1} = 10^n \cdot 10, \qquad (n+1)! = (n+1)n!$$

(To simplify ratios with exponents and factorials.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}$.

Simplify this:

$$\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \qquad \gg\gg \qquad \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\gg\gg \quad \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \quad \gg\gg \quad \frac{n^2+2n+1}{2n^2} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \dfrac{n^2+2n+1}{n^2} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ has ratio $R_n = \dfrac{n^2}{n^2+2n+1} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p-series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p-series.

09 Theory

Root Test (RooT)

Applicability: Any series.

Test Statement:

Suppose that $\sqrt[n]{|a_n|} \longrightarrow L$ as $n \to \infty$.

$\parallel$

$|a_n|^{1/n}$

Then:

$$\begin{aligned} L < 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely} \\ L > 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{diverges} \\ L = 1 \text{ or DNE}: &\quad \text{test inconclusive} \end{aligned}$$

Extra - Root test: explanation

$r = \frac{L+1}{2}$

The fact that $\sqrt[n]{|a_n|} \to L$ and $L < 1$ implies that eventually $\sqrt[n]{|a_n|} < r$ for all high enough n, where $r = \frac{L+1}{2}$ is the midpoint between L and 1.

Now, the equation $\sqrt[n]{|a_n|} < r$ is equivalent to the equation $|a_n| < r^n$.

Therefore, eventually the terms $|a_n|$ are each less than the corresponding terms of this convergent geometric series:

$$\sum_{n=1}^{\infty} r^n \;=\; 1 + r + r^2 + r^3 + \cdots$$

10 Illustration

Example - Root test examples

(a) Observe that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0 = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(b) Observe that $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(c) Observe that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^n$ converges because $\sqrt[n]{|a_n|} = \frac{3}{n} \to 0$ as $n \to \infty$.

Example - Ratio test versus root test

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

$\longrightarrow \frac{n^2}{5^2}\left(\frac{4}{5}\right)^n$

$\xrightarrow{1/n} \sqrt[n]{\frac{n^2}{5^2}} \cdot \frac{4}{5} \longrightarrow ??$

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \cdot \left(\frac{5}{4}\right)^n$$

$$\gg\gg \quad \frac{n^2+2n+1}{n^2} \cdot \frac{4}{5} \longrightarrow \frac{4}{5} < 1 \text{ as } n \to \infty$$

Therefore the series converges absolutely by the ratio test.

$\infty^{2/\infty}$ indeterminate form

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5} \longrightarrow ??$$

To compute the limit as $n \to \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln\left(\left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}\right) = \frac{2}{n}\ln\frac{n}{5} + \ln\frac{4}{5} \longrightarrow \ln 4/5$$

$\to 0$

so $e^{\ln 4/5} = 4/5$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln\frac{n}{5} \xrightarrow{d/dx} \frac{1}{n/5}\cdot\frac{1}{5}}{n/2 \xrightarrow{d/dx} 1/2} \qquad \gg\gg \qquad \frac{1/n}{1/2} \qquad \gg\gg \qquad \frac{2}{n} \longrightarrow 0 \text{ as } n \to \infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln\frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln\frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.