Calculus II - Lecture notes - W10

Series tests: strategy tips

Videos

Videos, Math Dr. Bob

- Series test round-up: Part I
- Series test round-up: Part II
- Series test round-up: Part III

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- How to choose a series convergence test

01 Theory

It can help to associate certain "strategy tips" to find convergence tests based on certain patterns.

Matching powers → Simple Divergence Test

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Use the SDT because we see the highest power is the *same* ($=1$) in numerator and denominator.

Rational or Algebraic → Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Use the LCT because we have a *rational or algebraic* function (positive terms).

Not rational, not factorials → Integral Test

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Use the IT because we do *not* have a rational/algebraic function, and we do *not* see factorials.

Rational, alternating → AST, and LCT or DCT

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$$

Use the AST because it's alternating. Then use the LCT (to find absolute convergence) because its a rational function.

Factorials → Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Use the RaT because we see a factorial. (In case of alternating + factorial, use RaT first.)

Recognize geometric → LCT or DCT

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Use the LCT or DCT comparing to $\frac{1}{3^n}$ because we see similarity to $\frac{1}{3^n}$ (recognize geometric).

Power series: Radius and Interval

Videos

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- Power series: Interval and Radius of Convergence
- Power series: Interval of Convergence Using Ratio Test
  - Further example
- Power series: Interval of Convergence Using Root Test
- Power series: Finding the Center

02 Theory

A power series looks like this:

$$f(x) \;=\; a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Power series are used to *build and study functions*. They allow a uniform "modeling framework" in which many functions can be described and compared. Power series are also convenient for *computers* because they provide a way to store and evaluate *differentiable* functions with numerical (approximate) values.

Small x needed for power series

The most important fact about power series is that they work for *small values of x*.

Many power series diverge for $|x|$ too big; but even when they converge, for big $|x|$ they converge more slowly, and partial sum approximations are less accurate.

The idea of a power series is a modification of the idea of a geometric series in which the common ratio r becomes a variable x, and each term has an additional *coefficient parameter* a_n controlling the relative contribution of different orders.

03 Theory

Every power series has a **radius of convergence** and an **interval of convergence**.

Radius of convergence

Consider a power series centered at $x = 0$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| \quad \gg\gg \quad \left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) |x|$$

Define the **radius of convergence** $R \in [0, \infty]$:

$$R \;=\; \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

Therefore:

$$\begin{aligned} |x| \; &< R \quad \implies \quad \text{converges} \\ |x| \; &> R \quad \implies \quad \text{diverges} \end{aligned}$$

We can build **shifted power series** for x near some other value c. Just replace the variable x with a shifted variable $u = x - c$:

$$a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots$$

$$\gg\gg \quad a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots$$

Now apply the ratio test to determine convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} |x-c|^{n+1}}{a_n |x-c|^n} \right| \quad \gg\gg \quad \left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) |x-c|$$

Define the **radius of convergence** $R \in [0, \infty]$:

$$R \;=\; \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

In the shifted setting, the radius of convergence limits the *distance from* :

$$\begin{aligned} |x-c| \; &< R \quad \implies \quad \text{converges} \\ |x-c| \; &> R \quad \implies \quad \text{diverges} \end{aligned}$$

Method:
To calculate the **interval of convergence** of a power series, follow these steps:

- Observe the center c of the shifted series (or $c = 0$ for no shift).

- Compute R using the limit of coefficient ratios.
- Write down the *preliminary interval* $(c-R,\ c+R)$.
- Plug each endpoint, $c-R$ and $c-R$, into the original series
    - Check for convergence
- Add in the *convergent endpoints*. (4 possible scenarios.)

04 Illustration

Example - Radius of convergence

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ (b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

(a) Ratio of terms:

$$\left| \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} \right| \quad \gg\gg \quad \frac{1/2^{n+1}}{1/2^n}|x| \quad \gg\gg \quad \frac{1}{2}|x|$$

Therefore $R=2$ and the series converges for $|x|<2$.

(b) This power series skips the odd powers. Apply the ratio test to just the even powers:

$$\left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right| \quad \gg\gg \quad \frac{(2n)!}{(2n+2)(2n+1)(2n)!}|x^2|$$

$$\gg\gg \quad \frac{1}{(2n+2)(2n+1)}|x^2|$$

$$\gg\gg \quad R=\infty$$

Example - Interval of convergence

Find the radius and interval of convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (b) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(1) Apply ratio test:

$$\left|\frac{\dfrac{(x-3)^{n+1}}{n+1}}{\dfrac{(x-3)^n}{n}}\right| \quad \gg\gg \quad \frac{n}{n+1}|x-3|$$

Therefore $R = 1$ and thus:

$$|x-3| < 1 \quad \Longrightarrow \quad \text{converges}$$

$$|x-3| > 1 \quad \Longrightarrow \quad \text{diverges}$$

Preliminary interval: $x \in (2,4)$.

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(2) Check endpoints:

Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\gg\gg \quad \text{converges by AST}$$

Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\gg\gg \quad \text{diverges as } p\text{-series}$$

Final interval of convergence: $x \in [2,4)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(1) Apply ratio test:

$$\left|\frac{\dfrac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\dfrac{(-3)^n x^n}{\sqrt{n+1}}}\right| \quad \gg\gg \quad \frac{|(-3)(-3)^n|}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{|(-3)^n|}|x|$$

$$\gg\gg \quad \frac{3\sqrt{n+1}}{\sqrt{n+2}}|x|$$

Therefore:

$$|x| < \frac{1}{3} \quad \Longrightarrow \quad \text{converges}$$

$$|x| > \frac{1}{3} \quad \Longrightarrow \quad \text{diverges}$$

Preliminary interval: $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

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(2) Check endpoints:

Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(-\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{diverges by LCT with } b_n = 1/\sqrt{n}$$

Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(+\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{converges by AST}$$

Final interval of convergence: $x \in \big(-1/3, 1/3\big]$

Exercise - Radius and interval

Find the radius and interval of convergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$

Interval of convergence - further examples

Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Ratio of terms:

$$\frac{n+1}{3^{n+2}} \cdot \frac{3^{n+1}}{n} |x+2| \quad \gg\gg \quad \frac{n+1}{3n} |x+2|$$

Therefore $R = 3$ and the preliminary interval is $x \in (-5, 1)$.

Check endpoints: $\displaystyle\sum \frac{n(-3)^n}{3^{n+1}}$ diverges and $\displaystyle\sum \frac{n(3)^n}{3^{n+1}}$ also diverges.

Final interval is $(-5, 1)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Ratio of terms:

$$\frac{\frac{1}{n+1}}{\frac{1}{n}}|4x+1| \quad \gg\gg \quad \frac{n}{n+1}|4x+1|$$

Therefore:

$$\begin{aligned} |4x+1| &< 1 \quad \iff \quad |x+1/4| < 1/4 \quad \implies \quad \text{converges} \\ |4x+1| &> 1 \quad \iff \quad |x+1/4| > 1/4 \quad \implies \quad \text{diverges} \end{aligned}$$

Preliminary interval: $x \in (0, 1/2)$

Check endpoints: $\sum \frac{(4 \cdot \frac{-1}{2} + 1)^n}{n}$ converges but $\sum \frac{1}{n}$ diverges.

Final interval of convergence: $[-1/2, 0)$

Power series as functions

Videos

Videos, Math Dr. Bob

- Power series functions: Derivative/Antiderivative - Basics
- Power series functions: Derivative/Antiderivative - Interval of Convergence
- Power series functions: Derivative/Antiderivative - More examples
- Power series functions: Geometric Power Series

05 Theory

Given a numerical value for x within the interval of convergence of a power series, the series sum may be considered as the output $f(x)$ of a function f.

Many techniques from algebra and calculus can be applied to such power series functions.

Addition and Subtraction:

$$\begin{aligned} f &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ g &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ \hline \\ f + g &= (a_0 + b_0) + (a_1 + b_1)\,x + (a_2 + b_2)\,x^2 + \cdots \end{aligned}$$

Summation notation:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \quad = \quad \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

Scaling:

$$cf \;=\; ca_0 + (ca_1)\,x + (ca_2)\,x^2 + \cdots$$

Summation notation:

$$c \sum_{n=0}^{\infty} a_n x^n \quad = \quad \sum_{n=0}^{\infty} (ca_n)\,x^n$$

Extra - Multiplication and composition

Multiplication:

$$f \cdot g = \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \cdot \left(b_0 + b_1 x + b_2 x^2 + \cdots\right)$$

$$= a_0 b_0 + \left(a_0 b_1 + a_1 b_0\right) x + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right) x^2 + \cdots$$

For example, suppose that the geometric power series $f(x) = 1 + x + x^2 + x^3 + \cdots$ converges, so $|x| < 1$. Then we have for its square:

$$\begin{aligned} f \cdot f \; = \; f(x)^2 \; &= \; (1 + x + x^2 + \cdots) \cdot (1 + x + x^2 + \cdots) \\ &= \; 1 + (1+1)x + (1+1+1)x^2 + \cdots \\ &= \; 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \; \sum_{n=0}^{\infty} (n+1) x^n \end{aligned}$$

Composition:

$$f(-x) \; = \; 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$\begin{aligned} f(2x^3) \; &= \; 1 + 2x^3 + (2x^3)^2 + \cdots \\ &= \; 1 + 2x^3 + 4x^6 + 8x^9 + \cdots \end{aligned}$$

Assume:

$$f \; = \; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \; = \; \sum_{n=0}^{\infty} a_n x^n$$

Then:

Differentiation:

$$\frac{df}{dx} \; = \; a_1 + (2a_2)\, x + (3a_3)\, x^2 + \cdots \quad = \quad \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Antidifferentiation:

$$\int f(x)\, dx \; = \; C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots \quad = \quad C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

For example, for the geometric series we have:

$$f = \; 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{df}{dx} = \; 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\int f\, dx = \; C + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \cdots$$

Do the series created with sums, products, derivatives etc., all converge? On what interval?

For the algebraic operations, the resulting power series will converge wherever both of the original series converge.

For calculus operations, the *radius* is preserved, but the *endpoints* are not necessarily:

Power series calculus - Radius preserved

If the power series $f(x)$ has radius of convergence R, then the power series $f'(x)$ and $\int f\,dx$ also have the same radius of convergence R.

Power series calculus - Endpoints not preserved

It is possible that a power series $f(x)$ converges at and endpoint a of its interval of convergence, yet f' and $\int f\,dx$ do *not* converge at a.

Extra - Proof of radius for derivative and integral series

Suppose $f(x)$ has radius of convergence R:

$$\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| \longrightarrow \frac{1}{R} \cdot |x| \quad \text{as } n \to \infty$$

Consider now the derivative f' and its ratios of successive terms:

$$\left|\frac{(n+1)a_{n+1}x^n}{na_nx^{n-1}}\right| = \left(\frac{n+1}{n}\right) \cdot \left|\frac{a_{n+1}}{a_n}\right| \cdot |x| \quad \overset{n\to\infty}{\longrightarrow} \quad 1 \cdot \frac{1}{R} \cdot |x| = \frac{1}{R} \cdot |x|$$

Consider instead the antiderivative $\int f\,dx$ and its ratios of successive terms:

$$\left|\frac{\left(\frac{1}{n+1}\right)a_nx^{n+1}}{\left(\frac{1}{n}\right)a_{n-1}x^n}\right| = \left(\frac{n}{n+1}\right) \cdot \left|\frac{a_n}{a_{n-1}}\right| \cdot |x| \quad \overset{n\to\infty}{\longrightarrow} \quad 1 \cdot \frac{1}{R} \cdot |x| = \frac{1}{R} \cdot |x|$$

In both these cases the ratio test provides that the series converges when $|x| < R$.

06 Illustration

Example - Geometric series: algebra meets calculus

Consider the geometric series as a power series functions:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Take the derivative of both sides of the *function*:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \ggg \quad \frac{1}{(1-x)^2} \quad \ggg \quad \left(\frac{1}{1-x}\right)^2$$

This means f satisfies the identity:

$$f' = f^2$$

Now compute the derivative of the *series*:

$$1 + x + x^2 + x^3 + \cdots \quad \overset{\frac{d}{dx}}{\ggg} \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

On the other hand, compute the square of the series:

$$\left(1 + x + x^2 + x^3 + \cdots\right)^2 \quad \ggg \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So we find that the *same relationship holds*, namely $f' = f^2$, for the closed formula and the series formula for this function.

Example - Manipulating geometric series: algebra

Find power series that represent the following functions:

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1+x^2}$ (c) $\dfrac{x^3}{x+2}$ (d) $\dfrac{3x}{2-5x}$

Solution

(a) $\dfrac{1}{1+x}$

(1) Rewrite in format $\frac{1}{1-u}$.

Introduce double negative:

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Choose $u = -x$.

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(2) Plug $u = -x$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

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(3) Plug in $u = -x$:

$$\gg\gg \qquad 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

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(4) Simplify:

$$\gg\gg \qquad 1 - x + x^2 - x^3 + \cdots$$

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(5) Final answer:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(b) $\dfrac{1}{1+x^2}$

(1) Rewrite in format $\frac{1}{1-u}$.

Rewrite:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Choose $u = -x^2$.

---

(2) Plug $u = -x^2$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = -x^2$:

$$\gg\gg \quad 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$\gg\gg \quad 1 - x^2 + x^4 - x^6 + \cdots$$

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(3) Final answer:

$$\frac{1}{1+x} = 1 - x^2 + x^4 - x^6 + \cdots$$

(c) $\dfrac{x^3}{x+2}$

(1) Rewrite in format $Ax^3 \cdot \frac{1}{1-u}$.

Rewrite:

$$\frac{x^3}{x+2} \qquad \gg\gg \qquad x^3 \cdot \frac{1}{2+x} \qquad \gg\gg \qquad x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$\gg\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1+\frac{x}{2}} \qquad \gg\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

Choose $u = -\frac{x}{2}$. Here $Ax^3 = \frac{1}{2}x^3$.

---

(2) Plug $u = -x^2$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = -\frac{x}{2}$:

$$\gg\gg \qquad 1 + (-\tfrac{x}{2}) + (-\tfrac{x}{2})^2 + (-\tfrac{x}{2})^3 + \cdots$$

$$\gg\gg \qquad 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

Obtain:

$$\frac{1}{1-\left(-\frac{x}{2}\right)} = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

---

(3) Multiply by $\frac{1}{2}x^3$.

Distribute:

$$\frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)} \qquad \ggg \qquad \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

Final answer:

$$\frac{x^3}{x+2} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

(d) $\dfrac{3x}{2-5x}$

(1) Rewrite in format $Ax \cdot \frac{1}{1-u}$.

Rewrite:

$$\frac{3x}{2-5x} \qquad \ggg \qquad 3x \cdot \frac{1}{2-5x}$$

$$\ggg \qquad 3x \cdot \frac{1}{2\left(1-\frac{5x}{2}\right)} \qquad \ggg \qquad \frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}}$$

Choose $u = \frac{5x}{2}$. Here $Ax = \frac{3}{2}x$.

---

(2) Plug $u = \frac{5x}{2}$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = \frac{5x}{2}$:

$$\ggg \qquad 1 + \left(\tfrac{5x}{2}\right) + \left(\tfrac{5x}{2}\right)^2 + \left(\tfrac{5x}{2}\right)^3 + \cdots$$

$$\ggg \qquad 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

Obtain:

$$\frac{1}{1-\frac{5x}{2}} = 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

---

(3) Multiply by $\frac{3}{2}x$.

Distribute:

$$\frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}} \qquad \ggg \qquad \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Final answer:

$$\frac{3x}{2-5x} = \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Example - Manipulating geometric series: calculus

Find a power series that represents $\ln(1+x)$.

Solution

(1) Differentiate to obtain similarity to geometric sum formula.

Differentiate $\ln(1+x)$:

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} \qquad \gg\gg \qquad \frac{1}{1-(-x)}$$

---

(2) Find power series of differentiated function.

Power series by modifying $\frac{1}{1-u}$ with $u=-x$:

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

---

(3) Integrate series to find original function.

Integrate both sides:

$$\int \frac{1}{1-(-x)}\,dx = \int 1 - x + x^2 - x^3 + x^4 - \cdots\,dx$$

$$\ln(1+x) = D + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Use known point to solve for D:

$$\ln(1+0) = D + 0 + 0 + \cdots \qquad \gg\gg \qquad 0 = D$$

Final answer:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Example - Recognizing and manipulating geometric series: Part I

(a) Evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.
(Hint: consider the series of $\ln(1-x)$.)

(b) Find a series approximation for $\ln(2/3)$.

Solution

(a)

(1) We know the series of $\frac{-1}{1-x}$:

$$\frac{-1}{1-x} = -(1 + x + x^2 + \cdots) = -1 - x - x^2 - \cdots$$

Notice that $\int \frac{-1}{1-x}\,dx = \ln(1-x) + C$; this is the desired function when $C = 0$.

Integrate the series term-by-term:

$$\int \frac{-1}{1-x}\,dx = \int -1 - x - x^2 - \cdots\,dx$$

$$\gg\!\gg \quad \ln(1-x) = D - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

Solve for D using $\ln(1-0) = 0$, so $0 = D - 0 - 0 - \cdots$ and thus $D = 0$. So:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad = \quad \sum_{n=1}^{\infty} -\frac{x^n}{n!}$$

---

(2) Notice the formula:

The series formula $\sum_{n=1}^{\infty} -\frac{x^n}{n!}$ looks similar to the formula $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$.

---

(3) Choose $x = -1$ to recreate the desired series:

We obtain equality by setting $x = -1$ because $-(-1)^n = (-1)^{n+1} = (-1)^{n-1}$.

Final answer is $\ln(1 - -1) = \ln 2$.

(b)

Find a series approximation for $\ln(2/3)$:

(1) Observe that $\ln(2/3) = \ln(1 - 1/3)$.

Therefore we can use the series $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

---

(2) Plug $x = 1/3$ into the series for $\ln(1-x)$.

Plug in and simplify:

$$\begin{aligned}\ln(2/3) = \ln(1-1/3) &= -1/3 - \frac{(1/3)^2}{2} - \frac{(1/3)^3}{3} - \cdots \\ &= -\frac{1}{3} - \frac{1}{3^2 \cdot 2} - \frac{1}{3^3 \cdot 3} - \cdots\end{aligned}$$

Example - Recognizing and manipulating geometric series: Part II

(a) Find a series representing $\tan^{-1}(x)$ using differentiation.

(b) Find a series representing $\int \frac{dx}{1+x^4}$.

Solution

(a)

(1) Notice that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

Obtain the series for $\frac{1}{1+x^2}$.

Let $u = -x^2$:

$$\frac{1}{1+x^2} \qquad \gg\gg \qquad \frac{1}{1-u} = 1 + u + u^2 + \cdots$$

$$\gg\gg \qquad 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

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(2) Integrate the series for $\frac{1}{1+x^2}$ by terms.

Set up the strategy. We know:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

and:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Integrate the series term-by-term:

$$\gg\gg \quad \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots\,dx$$

$$\gg\gg \quad D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Conclude:

$$\tan^{-1}(x) + C = D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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(3) Solve for $D - C$ by testing at $\tan^{-1}(0) = 0$.

Plug in:

$$\tan^{-1}(0) \;=\; D - C + 0 + \cdots + 0$$

$$\gg\gg \quad D - C = 0$$

Final answer: $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(b)

(1) Find a series representing the integrand.

Integrand is $\dfrac{1}{1+x^4}$.

Rewrite integrand in format of geometric series sum:

$$\frac{1}{1+x^4} \qquad \ggg \qquad \frac{1}{1-(-x^4)} \qquad \ggg \qquad \frac{1}{1-u}, \quad u=-x^4$$

Write the series:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$$

$$\ggg \quad 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

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(2) Integrate the series by terms:

$$\int 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \, dx \qquad \ggg \qquad C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots$$