W01 - Examples

Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution

(1) Define the cross section region.

Bounded above-right by $y = 6 - x$.

Bounded below-right by $y = 3x + 2$.

⚠ **These intersect at $x = 1$.**

Bounded at left by $x = 0$.

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(2) Define range of integration variable.

Rotated around y-axis, therefore use x for integration variable (shells!).

Integral over $x \in [0, 1]$:

$$V = \int_0^1 2\pi R h \, dr$$

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(3) Interpret R.

Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

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(4) Interpret h.

Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6 - x) - (3x + 2) \\ &= 4 - 4x \end{aligned}$$

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(5) Interpret dr.

dr is limit of Δr which equals Δx here so $dr = dx$.

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(6) Plug data in volume formula.

Insert data and compute integral:

$$V = \int_0^1 2\pi R h \, dr$$

$$= \int_0^1 2\pi \cdot x(4-4x)\, dx$$

$$= 2\pi \left(2x^2 - \frac{4x^3}{3}\right)\Big|_0^1 >= \frac{4\pi}{3}$$

A and T factors

Compute the integral: $\int x \cos x \, dx$

Solution

(1) Choose $u = x$.

Set $u(x) = x$ because x *simplifies* when differentiated.
(By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)

Remaining factor must be v':

$$v'(x) = \cos x$$

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(2) Compute u' and v.

Derive u:

$$u' = 1$$

Antiderive v':

$$v = \sin x$$

Obtain chart:

$$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \quad \begin{array}{ll} \int u \cdot > v' & \quad \text{original} \\ \int u' \cdot v & \quad \text{final} \end{array}$$

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(3) Plug into IBP formula.

Plug in all data:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

Compute integral on RHS:

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

Note: the *point* of IBP is that this integral is easier than the first one!

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(4) Final answer is: $x \sin x + \cos x + C$