W02 - Examples

Power product - odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

Solution

(1) Swap over the even bunch.

Max even bunch leaving power-one is $\sin^4 x$:

$$\sin^5 x \qquad \ggg \qquad \sin x \left(\sin^2 x\right)^2 \qquad \ggg \qquad \sin x \left(1 - \cos^2 x\right)^2$$

Apply to $\sin^5 x$ in the integrand:

$$\int \cos^2 x \cdot \sin^5 x \, dx \qquad \ggg \qquad \int \cos^2 x \cdot \sin x \left(1 - \cos^2 x\right)^2 dx$$

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(2) Perform u-substitution on the power-one integrand.

Set $u = \cos x$.

Hence $du = \sin x \, dx$. Recognize this in the integrand.

Convert the integrand:

$$\int \cos^2 x \cdot \sin x \big(1 - \cos^2 x\big)^2 dx \qquad \ggg \qquad \int \cos^2 x \cdot \big(1 \cos^2 x\big)^2 \big(\sin x \, dx\big)$$

$$\ggg \qquad \int u^2 \cdot (1 - u^2)^2 \, du$$

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(3) Perform the integral.

Expand integrand and use power rule to obtain:

$$\int u^2 \cdot (1 - u^2)^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

Insert definition $u = \cos x$:

$$\int \cos^2 x \cdot \sin^5 x \, dx \quad \ggg \quad \int u^2 \cdot (1 - u^2)^2 \, du$$

$$\ggg \quad \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

This is our final answer.

Power product - tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

Solution

(1) Try $du = \sec^2 x \, dx$.

Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^5 x \cdot \sec x \left(\sec^2 x \, dx \right)$$

We then must swap over remaining $\sec x$ into the $\tan x$ type.

Cannot do this because $\sec x$ has odd power. Need even to swap.

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(2) Try $du = \sec x \, \tan x \, dx$.

Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^4 x \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

Swap remaining $\tan x$ into $\sec x$ type:

$$\int (\tan^2 x)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

$$\gg\gg \qquad \int \left(\sec^2 x - 1 \right)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

Substitute $u = \sec x$ and $du = \sec x \, \tan x \, dx$:

$$\gg\gg \qquad \int (u^2 - 1)^2 \cdot u^2 \, du$$

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(3) Compute the integral in u and convert back to x.

Expand the integrand:

$$\gg\gg \qquad \int u^6 - 2u^4 + u^2 \, du$$

Apply power rule:

$$\gg\gg \qquad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

Plug back in, $u = \sec x$:

$$\gg\gg \qquad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

Trig sub in quadratic - completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}}$$

Solution

(1) Notice square root of a quadratic.

Complete the square to obtain Pythagorean form.

Find constant term for a complete square:

$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x-3)^2$$

Add and subtract desired constant term:

$$x^2 - 6x + 11 \qquad \ggg \qquad x^2 - 6x + 9 - 9 + 11$$

Simplify:

$$x^2 - 6x + 9 - 9 + 11 \qquad \ggg \qquad (x-3)^2 + 2$$

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(2) Perform shift substitution.

Set $u = x - 3$ as inside the square:

$$(x-3)^2 + 2 = u^2 + 2$$

Infer $du = dx$.

Plug into integrand:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}} \qquad \ggg \qquad \int \frac{du}{\sqrt{u^2 + 2}}$$

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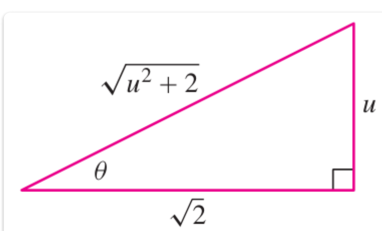
(3)

⚠ **Trig sub with $\tan\theta$.**

Identify triangle:



Use substitution $u = \sqrt{2}\tan\theta$. (From triangle or memorized tip.)

Infer $du = \sqrt{2}\sec^2\theta\,d\theta$.

Plug in data:

$$\int \frac{du}{\sqrt{u^2+2}} \qquad \gg\!\gg \qquad \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

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(4) Compute trig integral.

Use ad hoc formula:

$$\int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C$$

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(5) Convert trig back to x.

First in terms of u, referring to the triangle:

$$\tan\theta = \frac{u}{\sqrt{2}}, \qquad \sec\theta = \frac{\sqrt{u^2+2}}{\sqrt{2}}$$

Then in terms of x using $u = x - 3$.

Plug everything in:

$$\ln|\tan\theta + \sec\theta| + C \qquad \gg\!\gg \qquad \ln\left|\frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2+2}}{\sqrt{2}}\right| + C$$

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(6) Simplify using log rules.

Log rule for division gives us:

$$\ln\frac{f(x)}{a} = \ln f(x) - \ln a$$

The common denominator $\frac{1}{\sqrt{2}}$ can be pulled outside as $-\ln\sqrt{2}$.

The new term $-\ln\sqrt{2}$ can be “absorbed into the constant” (redefine C).

So we write our final answer thus:

$$\ln\left|x - 3 + \sqrt{(x-3)^2+2}\right| + C$$