W03 - Examples

Partial fractions with repeated factor

Find the partial fraction decomposition:

$$\frac{3x-9}{x^3+3x^2-4}$$

Solution

(1)

⚠ Check! Numerator is smaller than denominator (degree-wise).

Factor the denominator.

Rational roots theorem: $x=1$ is a zero.

Divide by $x-1$:

$$\frac{x^3+3x^2-4}{x-1}=x^2+4x+4$$

Factor again:

$$x^2+4x+4=(x+2)^2$$

Final factored form:

$$x^3+3x^2-4 \qquad \gg\gg \qquad (x-1)(x+2)^2$$

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(2) Write the generic PFD.

Allow all lower powers:

$$\frac{3x-9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

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(3) Solve for A, B, and C.

Multiply across by the common denominator:

$$3x-9=A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

For A, set $x=1$, obtain:

$$\begin{aligned} & 3\cdot 1-9=A(1+2)^2+B\cdot 0+C\cdot 0 \\ \gg\gg \quad & -6=9A \\ \gg\gg \quad & A=-2/3 \end{aligned}$$

For C, set $x=-2$, obtain:

$$\begin{aligned} & 3\cdot(-2)-9=A\cdot 0+B\cdot 0+C\cdot(-3) \\ \gg\gg \quad & -15=-3C \\ \gg\gg \quad & C=5 \end{aligned}$$

For B, insert prior results and solve.

Plug in A and C:

$$3x-9=-\frac{2}{3}(x+2)^2+B(x-1)(x+2)+5(x-1)$$

Now plug in another convenient x, say $x=3$:

$$0=-\frac{2}{3}\cdot 5^2+B\cdot 2\cdot 5+5\cdot 2$$

$$\frac{50}{3}-10=10B \qquad \gg\!\gg \qquad B=\frac{2}{3}$$

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(4) Plug in A, B, C for the final answer.

Final answer:

$$\frac{3x-9}{x^3+3x^2-4}=\frac{-2/3}{x-1}+\frac{2/3}{x+2}+\frac{5}{(x+2)^2}$$

Partial fractions - repeated quadratic, linear tops

Compute the integral:

$$\int \frac{x^3+1}{(x^2+4)^2}\,dx$$

Solution

(1) Compute the partial fraction decomposition.

Check that numerator degree is lower than denominator. ✓

Factor denominator completely. ✓ (No real roots.)

Write generic PFD:

$$\frac{x^3+1}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$$

Notice “linear over quadratic” in first term.

> ⚠ **Notice repeated factor: sum with incrementing powers up to 2.**

Common denominators and solve:

$$x^3+1=(Ax+B)(x^2+4)+Cx+D$$

$$\gg\!\gg \qquad x^3+1=Ax^3+Bx^2+(4A+C)x+4B+D$$

$$\gg\!\gg \qquad A=1,\ B=0$$

$$\gg\!\gg \qquad C=-4,\ D=1$$

Therefore:

$$\frac{x^3+1}{(x^2+4)^2}=\frac{x}{x^2+4}+\frac{-4x+1}{(x^2+4)^2}$$

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(2) Integrate by terms.

Integrate the first term using $u=x^2+4$:

$$\int \frac{x}{x^2+4}dx \quad \overset{u=x^2+4}{\gg\gg} \quad \frac{1}{2}\int \frac{du}{u}$$

$$\gg\gg \quad \frac{1}{2}\ln|u| + C \quad \gg\gg \quad \frac{1}{2}\ln\left|x^2+4\right| + C$$

Break up the second term:

$$\frac{-4x+1}{(x^2+4)^2} \quad \gg\gg \quad \frac{-4x}{(x^2+4)^2} + \frac{1}{(x^2+4)^2}$$

Integrate the first term of RHS:

$$\int \frac{-4x}{(x^2+4)^2}\,dx \quad \gg\gg \quad -2\int \frac{du}{u^2}$$

$$\gg\gg \quad \frac{2}{u} + C \quad \gg\gg \quad \frac{2}{x^2+4} + C$$

Integrate the second term of RHS:

$$\int \frac{dx}{(x^2+4)^2} \quad \overset{x=2\tan\theta}{\gg\gg} \quad \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta}$$

$$\gg\gg \quad \frac{1}{8}\int \cos^2\theta\,d\theta \quad \gg\gg \quad \frac{1}{16}\theta + \frac{1}{32}\sin(2\theta) + C$$

"Rationalize a quotient" - convert into PFD

Sometimes an integrand may be *converted* to a rational function using a *substitution*.

Consider this integral:

$$\int \frac{\sqrt{x+4}}{x}\,dx$$

Set $u = \sqrt{x+4}$, so $x = u^2 - 4$ and $dx = 2u\,du$:

$$\gg\gg \quad \int \frac{2u\,du}{u^2-4}$$

Now this rational function has a partial fraction decomposition:

$$\frac{2u}{u^2-4} \quad \gg\gg \quad \frac{2u}{(u-2)(u+2)} \quad \gg\gg \quad \frac{1}{u-2} + \frac{1}{u+2}$$

It is easy to integrate from there!

Exercise examples:

- To compute $\int \frac{\sqrt{x}}{x-1}\,dx$, try the substitution $u = \sqrt{x}$.
- To compute $\int \frac{dx}{\sqrt[2]{x}-\sqrt[3]{x}}$, try the substitution $u = \sqrt[6]{x}$.
- To compute $\int \frac{1}{x-\sqrt{x+2}}\,dx$, try the substitution $u = \sqrt{x+2}$.

Simpson's Rule on the Gaussian distribution

The function e^{x^2} is very important for probability and statistics, but it cannot be integrated analytically.

Apply Simpson's Rule to approximate the integral:

$$\int_0^1 e^{x^2}\,dx$$

with $\Delta x = 0.1$ and $n = 10$. What error bound is guaranteed for this approximation?

Solution

(1) We need a table of values of x_i and $y_i = f(x_i)$:

x_i :	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x_i)$:	$e^{0.0^2}$	$e^{0.1^2}$	$e^{0.2^2}$	$e^{0.3^2}$	$e^{0.4^2}$	$e^{0.5^2}$	$e^{0.6^2}$	$e^{0.7^2}$	$e^{0.8^2}$	$e^{0.9^2}$	$e^{1.0^2}$
$\approx$	1.000	1.010	1.041	1.094	1.174	1.284	1.433	1.632	1.896	2.248	2.718

These can be plugged into the Simpson Rule formula to obtain our desired approximation:

$$S_{10} = \frac{1}{3} \cdot 0.1 \cdot \Big(1.000 + 4 \cdot 1.010 + 2 \cdot 1.041 + 4 \cdot 1.094 + \cdots + 2 \cdot 1.896 + 4 \cdot 2.248 + 2.718\Big)$$

$$\approx 1.463$$

To find the error bound we need to find the smallest number we can manage for K_4.

Take four derivatives and simplify:

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

On the interval $x \in [0, 1]$, this function is maximized at $x = 1$. Use that for the optimal K_4:

$$f^{(4)}(1.000) = 206.589$$

Finally we plug this into the error bound formula:

$$\frac{K_4(b-a)^5}{180n^4} = \frac{206.589 \cdot 1.000^5}{180 \cdot 10^4} \approx 0.0001$$

$$\gg\!\gg \qquad \mathrm{Error}(S_{10}) \leq 0.0001$$