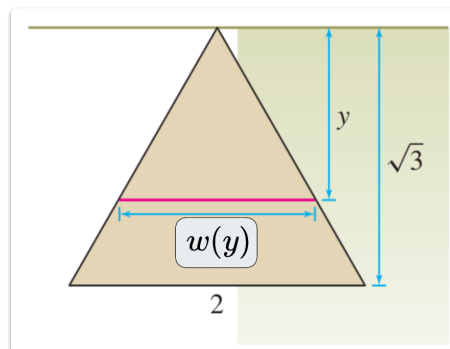


W05 - Examples

Hydrostatic force

Fluid force on a triangular plate

Find the total force on the submerged *vertical* plate with the following shape: Equilateral triangle, sides 2m, top vertex at the surface, liquid is oil with density $\rho = 900\text{kg/m}^3$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the vertex), and y increases going *down*.

Method 1: Geometry of similar triangles

Top triangle with base at $w(y)$ is similar to *total* triangle with base $\sqrt{3}$.

Therefore, corresponding parts have the same ratios.

Therefore:

$$\frac{w(y)}{y} = \frac{2}{\sqrt{3}}$$

$$\gg \gg w(y) = \frac{2}{\sqrt{3}}y$$

Method 2: Quick linear interpolation function

$$w(y) = 0 + \frac{2-0}{\sqrt{3}} \cdot y$$

$$\gg \gg w(y) = \frac{2}{\sqrt{3}}y$$

 Generalized “quick linear interpolation function” >

Generalization:

$$w(y) = \text{Edge}_1 + \frac{\text{Edge}_2 - \text{Edge}_1}{\text{Height}} (\pm y \pm a)$$

where:

- $\pm y$ is $+y$ when Edge_1 comes earlier (smaller y), and $-y$ if it comes later
- $\pm a$ is created to force the quantity $(\pm y \pm a)$ to equal 0 for the given y value at Edge_1

(2) Compute integral using width function:

Bounds:

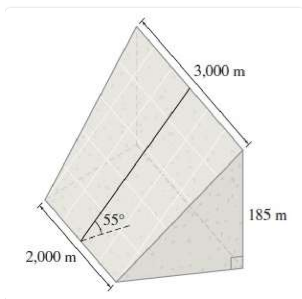
- $y = 0$ (shallow)
- $y = \sqrt{3}$ (deep)

Integral formula:

$$\begin{aligned} F &= \rho g \int_0^{\sqrt{3}} y w(y) dy \\ \ggg & 900 \cdot 9.8 \int_0^{\sqrt{3}} y \cdot 2y/\sqrt{3} dy \\ \ggg & 10184.5 \int_0^{\sqrt{3}} y^2 dy \\ \ggg & 10184.5 \frac{y^3}{3} \Big|_0^{\sqrt{3}} \ggg \mathbf{17640} \end{aligned}$$

Weight of water on a dam

Find the total hydrostatic force on an angled dam with the following geometric description: Tilted trapezoid. Base = 2,000m, Top = 3,000m, and *vertical* height 185m. The base is tilted at an angle of $\theta = 55^\circ$.



Solution

(1) Write the width function:

Establish coordinate system: $y = 0$ at water line (also the top edge), and increases going *down*.

“Quick linear interpolation function”:

$$w(y) = 3000 + \frac{2000 - 3000}{185}y$$

$$\gg \gg 3000 - \frac{1000}{185}y$$

(2) Incorporate angle of incline in strip thickness:

$$dz = \csc 55^\circ dy$$

So the area of a strip is:

$$dA = w(y) dz$$

$$\gg \gg \left(3000 - \frac{1000}{185}y \right) \csc 55^\circ dy$$

(3) Compute total force using integral formula.

Plug data into formula:

$$F = \rho g \int_a^b h(y) w(y) dz$$

$$\gg \gg \rho g \csc 55^\circ \int_0^{185} y \left(3,000 - \frac{1,000}{185}y \right) dy$$

$$\gg \gg 4.777 \times 10^{11}$$

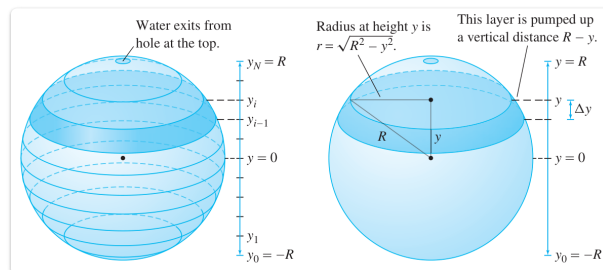
Work

Pumping water from spherical tank

Calculate the work done pumping water out of a spherical tank of radius $R = 5$ m.

Solution

(1) Slice the tank of water into horizontal layers:



Coordinate y is $y = 0$ at the center of the sphere, increasing upwards.

(2) Calculate weight of single slice:

$$\text{area of slice} = A(y) = \pi r^2 \ggg \pi(5^2 - y^2)$$

$$\text{volume of slice} = dV = A(y) dy \ggg \pi(5^2 - y^2) dy$$

$$\text{weight of slice} = dF = \rho g dV = \rho g \pi(5^2 - y^2) dy$$

(3) Work to lift out single slice:

Distance to raise a slice:

$$h(y) = 5 - y$$

Then:

$$\text{work to lift out slice} = dW = h(y) dF = \rho g \pi h(y) (5^2 - y^2) dy$$

(4) Total work by integrating dW over all slices:

$$\begin{aligned} \int dW &\ggg \int_{-5}^{+5} \left(9800 \frac{\text{kg}}{\text{m}^3 \text{s}^2} \right) \pi (5^2 - y^2) (5 - y) dy && \text{(Note A)} \\ &\ggg \approx 2.6 \times 10^7 \text{ J} \end{aligned}$$

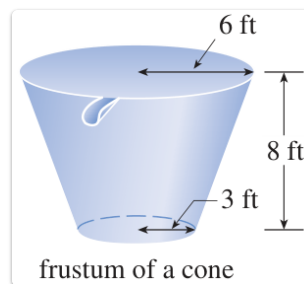
Note A: The integration runs over all slices, which start at $y = -5$ (bottom of tank), and end at $y = +5$ (top of tank).

Question: Extra question: what if the spigot sits 2m above the tank?

Question: Extra question: what if the tank starts at just 3m of water depth?

Water pumped from a frustum

Find the work required to pump water out of the frustum in the figure. Assume the weight of water is $\rho = 62.5 \text{ lb/ft}^3$.



Solution

(1) Find weight of a horizontal slice.

Coordinate $y = 0$ at top, increasing downwards.

Use $r(y)$ for radius of cross-section circle.

Linear decrease in r from $r(0) = 6$ to $r(8) = 3$:

$$r(y) = 6 - \frac{3}{8}y$$

Area is πr^2 :

$$\text{Area}(y) = \pi \left(6 - \frac{3}{8}y\right)^2$$

Weight = density $\times$ area $\times$ thickness:

$$\text{weight of layer} = \rho \pi \left(6 - \frac{3}{8}y\right)^2 dy$$

(2) Find work to pump out a horizontal layer.

Layer at y is raised a distance of y .

Work to raise layer at y :

$$\rho \pi y \left(6 - \frac{3}{8}y\right)^2 dy$$

(3) Integrate over all layers.

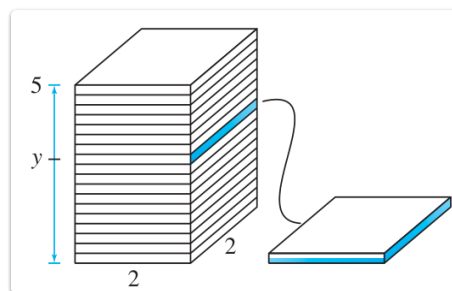
Integrate from top to bottom of frustum:

$$\begin{aligned} \int_0^8 \rho \pi y \left(6 - \frac{3}{8}y\right)^2 dy &= 528\pi\rho \\ &= 528\pi \cdot 62.5 \\ &\approx 1.04 \times 10^5 \text{ ft-lb} \end{aligned}$$

Final answer is 1.04×10^5 ft-lb.

Raising a building

Find the work done to raise a cement columnar building of height 5 m and square base 2 m per side. Cement has a density of 1500 kg/m^3 .



Solution

(1) Weight of each layer:

$$dV = A(y) dy \ggg 4 dy$$

$$dM = \rho dV \ggg 1500 \cdot 4 dy$$

$$dF = g dM \ggg 9.8 \cdot 6000 dy$$

(2) Work to lift layer into place:

$$dW = \text{weight} \times \text{distance raised} \ggg y \cdot 58800 dy$$

(3) Find total work as integral over the layers:

$$\begin{aligned} W &= \int dW \ggg \int_0^5 58800 y dy \\ &\ggg 735 \text{ kJ} \end{aligned}$$

Raising a chain

An 80 ft chain is suspended from the top of a building. Suppose the chain has weight density 0.5 lb/ft. What is the total work required to reel in the chain?

Solution

(1) Compute weight of a 'link' (vertical slice of the chain):

$$\begin{aligned} dF &= \text{density} \times \text{length} \\ &= 0.5 dy \end{aligned}$$

(2) Work dW to raise link to top:

Each link (slice) is raised from height y to height 80:

$$h(y) = (80 - y) \text{ ft}$$

Then:

$$dW = (80 - y) \cdot 0.5 dy$$

(3) Integrate over the chain for total work:

$$\begin{aligned} \int dW &\ggg \int_0^{80} (80 - y) \cdot 0.5 dy \\ &\ggg 1600 \text{ ft-lb} \end{aligned}$$

Raising a leaky bucket

Suppose a bucket is hoisted by a cable up an 80 ft tower. The bucket is lifted at a constant rate of 2 ft/sec and is leaking water weight at a constant rate of 0.2 lb/sec. The initial weight of water is 50 lb. What is the total work performed against gravity in lifting the water? (Ignore the bucket itself and the cable.)

Solution

(1) Compute total force from water $F(y)$:

Choose coordinate $y = 0$ at base, $y = 80$ at top.

Rate of water weight loss per unit height:

$$\begin{aligned} \frac{\text{rate of leak}}{\text{rate of lift}} &= \text{leaked weight per foot} \\ \gg \gg \frac{0.2 \text{ lb/sec}}{2 \text{ ft/sec}} &\gg \gg 0.1 \text{ lb/ft} \end{aligned}$$

Total water weight at height y :

$$F(y) = (50 - 0.1y) \text{ lb}$$

(2) Work to raise bucket by dy :

$$dW = F(y) dy \gg \gg (50 - 0.1y) dy$$

(3) Total work by integrating dW :

$$\begin{aligned} W &= \int_a^b dW \gg \gg \int_a^b (50 - 0.1y) dy \\ &\gg \gg 50y - 0.05y^2 \Big|_0^{80} = 3680 \text{ ft-lb} \end{aligned}$$

△ Change of method and integral formula!

For this example, we use the formula $\int F(y) dy$ rather than the formula $\int h(y) dF$ used in the earlier examples.

- This integral sums over the work dW to lift macroscopic material through each microscopic dy as if in sequence, and dy thus represents *distance lifted*.
- Earlier examples summed over the work dW to lift microscopic material through the macroscope $h(y)$ (all the way up).