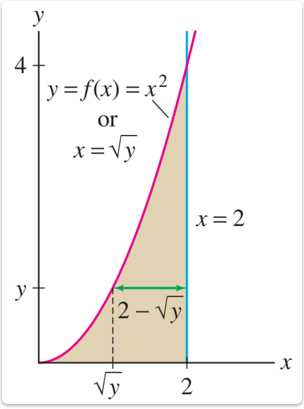
W06 - Examples

Moments and center of mass

CoM of a parabolic plate

Find the CoM of the region depicted:



Solution

(1) Compute the total mass:

Area under the curve with density factor ρ:

$$M = \int_0^2 \rho\, x^2\, dx \quad \ggg \quad \rho\, \frac{x^3}{3}\bigg|_0^2 \quad \ggg \quad \frac{8\rho}{3}$$

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(2) Compute M_y:

Formula:

$$M_y = \int_a^b \rho\, x\, dA$$

Interpret and calculate:

$$M_y = \int_0^2 \rho\, x\, f(x)\, dx \quad \ggg \quad \rho \int_0^2 x^3\, dx$$

$$\ggg \quad 4\rho = M_y$$

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(3) Compute M_x:

Formula:

$$M_x = \int_c^d \rho\, y\, dA$$

Width of horizontal strips between the curves:

$$w(y) = 2 - \sqrt{y}$$

Interpret dA:

$$dA = (2 - \sqrt{y})\, dy$$

Calculate integral:

$$M_x = \int_c^d \rho\, y\, dA \quad \gg\!\gg \quad \int_0^4 \rho\, y(2-\sqrt{y})\, dy$$

$$\gg\!\gg \quad \int_0^4 \rho\, y(2-\sqrt{y})\, dy \quad \gg\!\gg \quad \int_0^4 \rho\, 2y\, dy - \int_0^4 \rho\, y^{3/2}\, dy$$

$$\gg\!\gg \quad \frac{16\rho}{5} = M_x$$

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(4) Compute CoM coordinates from moments:

CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

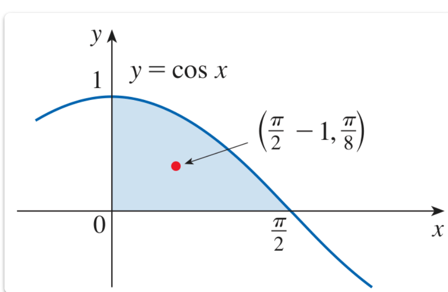
Insert data:

$$\bar{x} = \frac{4\rho}{8\rho/3} \quad \gg\!\gg \quad \frac{3}{2}, \qquad \bar{y} = \frac{16\rho/5}{8\rho/3} \quad \gg\!\gg \quad \frac{6}{5}$$

$$\text{CoM} \;=\; (\bar{x}, \bar{y}) = \boxed{\left(\frac{3}{2}, \frac{6}{5}\right)}$$

Computing CoM using only vertical strips

Find the CoM of the region:



Solution

(1) Compute the total mass M:

Area under the curve times density ρ:

$$\int_0^{\pi/2} \rho\, \cos x\, dx = \rho \sin x\Big|_0^{\pi/2} = \rho$$

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(2) Compute M_y using vertical strips:

Plug $f(x) = \cos x$ into formula:

$$M_y = \int_a^b \rho\, x\, f(x)\, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho\, x \cos x\, dx$$

Integration by parts. Set $u = x$, $v' = \cos x$ and so $u' = 1$, $v = \sin x$:

$$\int_0^{\pi/2} \rho\, x \cos x\, dx \quad \gg\gg \quad \rho\, x \sin x\Big|_0^{\pi/2} - \rho \int_0^{\pi/2} \sin x\, dx$$

$$\gg\gg \qquad \frac{\pi\rho}{2} \cdot 1 - \rho\left(-\cos\frac{\pi}{2} - -\cos 0\right) = \rho\left(\frac{\pi}{2} - 1\right)$$

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(3) Compute M_x, also using vertical strips:

Plug $f_2(x) = f(x) = \cos x$ and $f_1(x) = 0$ into formula:

$$M_x = \int_0^{\pi/2} \rho\, \tfrac{1}{2} f_2^2\, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho\, \tfrac{1}{2} \cos^2 x\, dx$$

Integration by 'power to frequency conversion':

$$\int_0^{\pi/2} \rho\, \tfrac{1}{2} \cos^2 x\, dx \quad \gg\gg \quad \frac{\rho}{4} \int_0^{\pi/2} (1 + \cos 2x)\, dx$$

$$\gg\gg \qquad \frac{\rho}{4} x\Big|_0^{\pi/2} + \frac{\rho \sin 2x}{8}\Big|_0^{\pi/2} = \frac{\pi\rho}{8}$$

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(4) Compute CoM:

CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Plug in data:

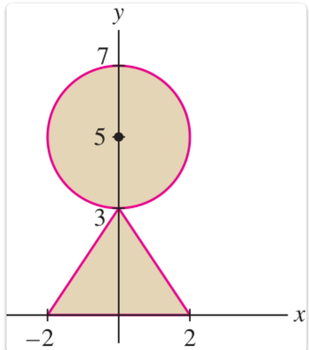
$$\bar{x} = \frac{\rho(\pi/2 - 1)}{\rho} \quad \gg\gg \quad \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{\pi\rho/8}{\rho} \quad \gg\gg \quad \frac{\pi}{8}$$

$$\text{CoM} \;=\; (\bar{x}, \bar{y}) \;=\; \boxed{\left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)}$$

Center of mass using moments and symmetry

Find the center of mass of the two-part region:



Solution

(1) Symmetry: CoM on y-axis

Because the region is symmetric in the y-axis, the CoM must lie on that axis. Therefore $\bar{x} = 0$.

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(2) Additivity of moments:

Write M_x for the total x-moment (distance measured to the x-axis from above).

Write M_x^{tri} and M_x^{circ} for the x-moments of the triangle and circle.

Additivity of moments equation:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}}$$

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(3) Find moment of the circle M_x^{circ}:

By symmetry we know $\bar{x}^{\text{circ}} = 0$.

By symmetry we know $\bar{y}^{\text{circ}} = 5$.

Area of circle with $r = 2$ is $A = 4\pi$, so total mass is $M = 4\pi\rho$.

Centroid-from-moments equation:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M}$$

Solve the equation for M_x^{circ}:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M} \quad \ggg \quad 5 = \frac{M_x^{\text{circ}}}{4\pi\rho}$$

$$\ggg \quad M_x^{\text{circ}} = 20\pi\rho$$

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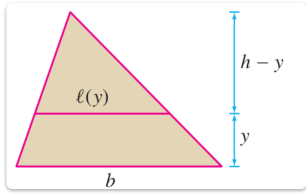
(4) Find moment of the triangle M_x^{tri} using integral formula:

Similar triangles:



Quick linear interpolation function:

$$\ell(y) \;=\; 0 + \frac{b-0}{h}(-y+h)$$

$$\gg\gg \quad \ell(y) \;=\; b - \frac{b}{h}y$$

Thus:

$$M_x^{\text{tri}} = \rho \int_0^h y\,\ell(y)\,dy \quad \gg\gg \quad \rho \int_0^h y\left(b - \frac{b}{h}y\right)dy$$

$$\gg\gg \quad \rho\left(\frac{by^2}{2} - \frac{by^3}{3h}\right)\Big|_0^h \quad \gg\gg \quad \frac{\rho b h^2}{6}$$

Conclude:

$$M_x^{\text{tri}} \quad \gg\gg \quad \frac{\rho b h^2}{6} \quad \gg\gg \quad \frac{\rho 4 \cdot 3^2}{6} \quad \gg\gg \quad 6\rho$$

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(5) Apply additivity:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}} \quad \gg\gg \quad \rho(20\pi + 6)$$

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(6) Total mass of region:

Area of circle is 4π. Area of triangle is $\frac{1}{2} \cdot 4 \cdot 3 = 6$. Thus $M = \rho A = \rho(4\pi + 6)$.

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(7) Compute center of mass $\bar{y}$ from total M_x and total M:

We have $M_x = \rho(20\pi + 6)$ and $M = \rho(4\pi + 6)$. Plug into formula:

$$\bar{y} = \frac{M_x}{M} \quad \gg\gg \quad \frac{\rho(20\pi + 6)}{\rho(4\pi + 6)} \approx 3.71$$

$$\text{CoM} \;=\; (\bar{x}, \bar{y}) = \boxed{(0, 3.71)}$$

Improper integrals

Improper integral - infinite bound

Show that the improper integral $\int_2^\infty \frac{dx}{x^3}$ converges. What is its value?

Solution

(1) Replace infinity with a new symbol R:

Compute the integral:

$$\int_2^R \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\Big|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

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(2) Take limit as $R \to \infty$:

$$\lim_{R\to\infty} \frac{1}{8} - \frac{1}{2R^2} = \frac{1}{8}$$

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(3) Improper integral definition:

$$\int_2^\infty \frac{dx}{x^3} \quad \gg\gg \quad \lim_{R\to\infty}\int_2^R \frac{dx}{x^3} \quad \gg\gg \quad \frac{1}{8}$$

Therefore that $\int_2^\infty \frac{dx}{x^3}$ converges and equals $1/8$.

Improper integral - infinite integrand

Show that the improper integral $\int_0^9 \frac{dx}{\sqrt{x}}$ converges. What is its value?

Solution

(1) Replace the 0 where $\frac{1}{\sqrt{x}}$ diverges with a new symbol a:

$$\int_a^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad \int_a^9 x^{-1/2}\,dx$$

$$\gg\gg \quad 2x^{+1/2}\Big|_a^9 \quad \gg\gg \quad 6 - 2\sqrt{a}$$

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(2) Take limit as $a \to 0^+$:

$$\lim_{a\to 0^+} 6 - 2\sqrt{a} = 6$$

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(3) Improper integral definition:

$$\int_a^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad \lim_{a\to 0^+}\int_a^9 \frac{dx}{\sqrt{x}} \quad \gg\gg \quad 6$$

Conclude that $\int_0^9 \frac{dx}{\sqrt{x}}$ converges to 6.

Improper integral - infinity inside the interval

Does the integral $\displaystyle\int_{-1}^{+1} \frac{1}{x}\,dx$ converge or diverge?

Solution

(1) WRONG APPROACH:

It is *tempting* to compute the integral *incorrectly*, like this:

$$\int_{-1}^{+1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{+1} = \ln|2| - \ln|-2| = 0$$

But this is wrong. There is an infinite integrand at $x = 0$. We must instead break it into parts.

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(2) Identify discontinuity (infinity) at $x = 0$:

$$\int_{-1}^{+1} \frac{1}{x}\,dx \quad \gg\gg \quad \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{+1} \frac{1}{x}\,dx$$

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(3) Improper integral definition:

$$\gg\gg \quad \lim_{R\to 0^-} \int_{-1}^{R} \frac{1}{x}\,dx + \lim_{R\to 0^+} \int_{R}^{+1} \frac{1}{x}\,dx$$

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(3) Integrate:

$$\int_{-1}^{R} \frac{1}{x}\,dx \quad \gg\gg \quad \ln|R| - \ln|-1| \quad \gg\gg \quad \ln|R|$$

$$\int_{R}^{+1} \frac{1}{x}\,dx \quad \gg\gg \quad \ln|1| - \ln|R| \quad \gg\gg \quad -\ln R$$

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(4) Take limits:

$$\lim_{R\to 0^-} \ln|R| = -\infty, \qquad \lim_{R\to 0^+} -\ln R = +\infty$$

Neither limit is finite. For $\displaystyle\int_{-1}^{+1} \frac{1}{x}\,dx$ to exist we'd need *both* of these limits to be finite. So: the original integral diverges.