

W07 - Examples

Sequences

Geometric sequence revealing the format

Find a_0 and r and a_n (written in the geometric sequence format) for the following geometric sequences:

(a) $a_n = \left(-\frac{1}{2}\right)^n$

(b) $b_n = -3 \left(\frac{2^{n+1}}{5^n}\right)$

(c) $c_n = e^{5+7n}$

Solution

(a)

Plug in $n = 0$ to obtain $a_0 = 1$. Notice that $a_{n+1}/a_n = -1/2$ and so therefore $r = -1/2$. Then the 'general term' is $a_n = a_0 \cdot r^n = 1 \cdot (-1/2)^n$.

(b)

Rewrite the fraction:

$$\frac{2^{n+1}}{5^n} \gg \gg 2 \cdot \left(\frac{2}{5}\right)^n$$

Plug that in and observe $b_n = -6 \cdot (2/5)^n$. From this format we can *read off* $b_0 = -6$ and $r = 2/5$.

(c)

Rewrite:

$$c_n \gg \gg e^5 \cdot e^{7n} \gg \gg e^5 \cdot (e^7)^n$$

From this format we can *read off* $c_0 = e^5$ and $r = e^7$.

Series

Geometric series - total sum and partial sums

The geometric series *total sum* S can be calculated using a "*shift technique*" as follows:

(1) Compare S and rS :

$$\begin{array}{rcl} S & = & a_0 + a_0r + a_0r^2 + a_0r^3 + \dots \\ \gg \gg \times r & & \\ rS & = & a_0r + a_0r^2 + a_0r^3 + a_0r^4 + \dots \end{array}$$

(2) Subtract second line from first line, many cancellations:

$$\begin{array}{rcl}
 S & = & a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \dots \\
 - (rS & = & a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \dots) \\
 \hline
 S - rS & = & a_0
 \end{array}$$

(3) Solve to find S :

$$S = \frac{a_0}{1-r}$$

△ Note: this calculation *assumes* that S exists, i.e. that the series *converges*.

The geometric series *partial sums* can be calculated similarly, as follows:

(1) Compare S and rS :

$$\begin{array}{rcl}
 S_N & = & a_0 + a_0 r + a_0 r^2 + \dots + a_0 r^N \\
 \xrightarrow{\times r} rS_N & = & a_0 r + a_0 r^2 + \dots + a_0 r^N + a_0 r^{N+1}
 \end{array}$$

(2) Subtract second line from first line, many cancellations:

$$\begin{array}{rcl}
 S_N & = & a_0 + a_0 r + a_0 r^2 + \dots + a_0 r^N \\
 - (rS_N & = & a_0 r + a_0 r^2 + \dots + a_0 r^N + a_0 r^{N+1}) \\
 \hline
 S_N - rS_N & = & a_0 - a_0 r^{N+1}
 \end{array}$$

(3) Solve to find S_N :

$$\begin{aligned}
 S_N &= a_0 \frac{1 - r^{N+1}}{1 - r} \\
 &= \frac{a_0}{1 - r} - \frac{a_0}{1 - r} r^{N+1} = S - S r^{N+1}
 \end{aligned}$$

(4) The last formula is revealing in its own way. Here is what it means in terms of terms:

$$\begin{aligned}
 a_0 + a_0 r + \dots + a_0 r^N &= \\
 &= a_0 + a_0 r + a_0 r^2 + \dots \\
 &\quad - (a_0 r^{N+1} + a_0 r^{N+2} + \dots)
 \end{aligned}$$

Convergence

L'Hopital's Rule for sequence limits

- (a) What is the limit of $a_n = \frac{\ln n}{n}$?
- (b) What is the limit of $b_n = \frac{(\ln n)^2}{n}$?
- (c) What is the limit of $c_n = n \left(\sqrt{n^2 + 1} - \sqrt{n} \right)$?

Solution

(a)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} \xrightarrow{\frac{\frac{1}{x}}{1}} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$$

(b)

Identify indeterminate form $\frac{\infty}{\infty}$. Change from n to x and apply L'Hopital:

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} \quad \gg \gg \quad \lim_{x \rightarrow \infty} \frac{2 \ln x \cdot \frac{1}{x}}{1} = 2 \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad (\text{by } a_n \text{ result}) = 0$$

(c)

(1) Identify form $\infty \cdot 0$ and rewrite as $\frac{\infty}{\infty}$:

$$n \left(\sqrt{n^2 + 1} - \sqrt{n} \right) \quad \gg \gg \quad \frac{\sqrt{n^2 + 1} - \sqrt{n}}{1/n}$$

(2) Change from n to x and apply L'Hopital:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1} - \sqrt{x}}{1/x} \quad \gg \gg \quad \frac{\frac{1}{2}(x^2 + 1)^{-1/2}(2x) - \frac{1}{2}x^{-1/2}}{-1/x^2}$$

(3) Simplify:

$$\gg \gg \quad \frac{-2x^3}{\sqrt{x^2 + 1}} + x^{3/2} = \frac{-2x^3 + x^{3/2}\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$$

(4) Consider the limit:

$$\frac{-2x^3 + x^{3/2}\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \xrightarrow{x \rightarrow \infty} \frac{-2x^3 + x^{3/2}x}{x} \rightarrow \frac{-2x^3}{x} \rightarrow -\infty$$

Squeeze theorem

Use the squeeze theorem to show that $\frac{4^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$.

Solution

(1) We will squeeze the given general term above 0 and below a sequence b_n that we must devise:

$$0 \leq \frac{4^n}{n!} \leq b_n$$

(2) We need b_n to satisfy $b_n \rightarrow 0$ and $\frac{4^n}{n!} \leq b_n$. Let us study $\frac{4^n}{n!}$.

$$\frac{4^n}{n!} = \frac{4 \cdot 4 \cdot \dots \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{n(n-1) \dots 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

(3) Now for the trick. Collect factors in the middle bunch:

$$\frac{4^n}{n!} = \frac{4}{n} \left(\frac{4}{n-1} \cdot \frac{4}{n-2} \cdot \dots \cdot \frac{4}{7} \cdot \frac{4}{6} \cdot \frac{4}{5} \right) \frac{4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1}$$

(4) Each factor in the middle bunch is < 1 so the entire middle bunch is < 1 . Therefore:

$$\frac{4^n}{n!} < \frac{4}{n} \cdot \frac{4^4}{4!} = \frac{1024}{24n}$$

Now we can easily see that $1024/24n \rightarrow 0$ as $n \rightarrow \infty$, so we set $b_n = 1024/24n$ and we are done.

Monotonicity

Show that $a_n = \sqrt{n+1} - \sqrt{n}$ converges.

Solution

(1) Observe that $a_n > 0$ for all n .

Because $n + 1 > n$, we know $\sqrt{n+1} > \sqrt{n}$.

Therefore $\sqrt{n+1} - \sqrt{n} > 0$

(2) Change n to x and show a_x is decreasing.

New formula: $a_x = \sqrt{x+1} - \sqrt{x}$ considered as a *differentiable* function.

△ Take derivative to show decreasing.

Derivative of a_x :

$$\frac{d}{dx} a_x = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

(3) Simplify:

$$\gg \gg \frac{2(\sqrt{x} - \sqrt{x+1})}{4\sqrt{x}\sqrt{x+1}}$$

Denominator is > 0 . Numerator is < 0 . So $\frac{d}{dx} a_x < 0$ and a_x is monotone decreasing.

Therefore a_n is monotone decreasing as $n \rightarrow \infty$.