W08 - Examples

Simple divergence test examples

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

- This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

- This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.
- We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \ldots,$" but that is not a limit value.

p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p \leq 1$

Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge?

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n \ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

(1) The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.

Check applicability.

Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty)$.

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(2) Apply the integral test to $f(x)$.

Integrate $f(x)$:

$$\int_2^\infty \frac{1}{x \ln x}\, dx \quad \gg\!\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u}\, du$$

$$\gg\!\gg \quad \lim_{R\to\infty} \ln u\Big|_{\ln 1}^{R} \quad \gg\!\gg \quad \infty$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n \ln n}$ *diverges*.

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(3) Apply the integral test to $g(x)$.

Integrate $g(x)$:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx \quad \gg\!\gg \quad \int_{u=\ln 2}^\infty \frac{1}{u^2}\, du$$

$$\gg\!\gg \quad \lim_{R\to\infty} -u^{-1}\Big|_{\ln 2}^{R} \quad \gg\!\gg \quad \frac{1}{\ln 2}$$

Conclude: $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ *converges*.