W09 - Examples

Direct comparison test rational functions

(a) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ *converges* by the DCT.

Choose: $a_n = \frac{1}{\sqrt{n}\,3^n}$ and $b_n = \frac{1}{3^n}$

Check: $0 < \frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$

Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

---

(b) The series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ *converges* by the DCT.

Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.

Check: $0 \leq \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$

Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

(c) The series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ *converges* by the DCT.

Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$

Check: $0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)

Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

(d) The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ *diverges* by the DCT.

Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$

Check: $0 \leq \frac{1}{n} \leq \frac{1}{n-1}$

Observe: $\sum \frac{1}{n}$ is a divergent p-series

Limit comparison test examples

(a) The series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ *converges* by the LCT.

Choose: $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty} \frac{2^n}{2^n - 1} \quad \ggg \quad 1 \;=:\; L$$

Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

---

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ *diverges* by the LCT.

Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = n^{-1/2}$

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \xrightarrow{n\to\infty} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2 \;=:\; L$$

Observe: $\sum n^{-1/2}$ is a divergent p-series

(c) The series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ *converges* by the LCT.

Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = n^{-2}$

Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \ggg \quad \lim_{n\to\infty} \frac{n^4}{n^4 - n - 1} \quad \ggg \quad 1 \;=:\; L$$

Observe: $\sum_{n=2}^{\infty} n^{-2}$ is a converging p-series

Alternating series test basic illustration

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.

Therefore the first series converges *conditionally*.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converges by the AST.

Notice the funny notation: $\cos n\pi = (-1)^n$.

This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Approximating pi

The Taylor series for $\tan^{-1} x$ is given by:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate π with an error less than 0.001.

Solution

(1) The main idea is to use $\tan \frac{\pi}{4} = 1$ and thus $\tan^{-1} 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

---

(2) Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. Therefore, calculate n such that $a_{n+1} < 0.001$, and then we will know:

$$|E_n| < a_{n+1} < 0.001$$

---

(3) The general term is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Now solve:

$$a_{n+1} = \frac{4}{2n+1} < 0.001$$

$$\gg\gg \qquad \frac{4}{0.001} < 2n + 1$$

$$\gg\gg \qquad 3999 < 2n$$

$$\gg\gg \qquad 2000 \leq n$$

We conclude that at least 2000 terms are necessary to be confident (by the error formula) that the approximation of π is accurate to within 0.001.

Ratio test examples

(a) Observe that $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n+1}$ and thus $R_n \to 0 = L < 1$. Therefore the RaT implies that this series converges.

Simplify the ratio:

$$\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{n!}{10^n}} \quad \gg\gg \quad \frac{(n+1)!}{10^{n+1}} \cdot \frac{n!}{10^n}$$

$$\gg\gg \quad \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \quad \gg\gg \quad \frac{10}{n+1} \overset{n\to\infty}{\longrightarrow} 0$$

Notice this technique! We *frequently* use these rules:

$$10^{n+1} = 10^n \cdot 10, \qquad (n+1)! = (n+1)n!$$

(To simplify ratios with exponents and factorials.)

(b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}$.

Simplify this:

$$\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \quad \gg\gg \quad \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\gg\gg \quad \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \quad \gg\gg \quad \frac{n^2+2n+1}{2n^2} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \dfrac{n^2+2n+1}{n^2} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has ratio $R_n = \dfrac{n^2}{n^2+2n+1} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p-series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p-series.

Root test examples

(a) Observe that $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0 = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(b) Observe that $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(c) Observe that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^n$ converges because $\sqrt[n]{|a_n|} = \frac{3}{n} \to 0$ as $n \to \infty$.

Ratio test versus root test

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$\left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \cdot \left(\frac{5}{4}\right)^n$$

$$\ggg \qquad \frac{n^2+2n+1}{n^2}\cdot\frac{4}{5} \longrightarrow \frac{4}{5} < 1 \text{ as } n\to\infty$$

Therefore the series converges absolutely by the ratio test.

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}$$

To compute the limit as $n \to \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln\left(\left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}\right) = \frac{2}{n}\ln\frac{n}{5} + \ln\frac{4}{5}$$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln\frac{n}{5} \overset{d/dx}{\longrightarrow} \frac{1}{n/5}\cdot\frac{1}{5}}{n/2 \overset{d/dx}{\longrightarrow} 1/2} \qquad \ggg \qquad \frac{1/n}{1/2} \qquad \ggg \qquad \frac{2}{n} \longrightarrow 0 \text{ as } n\to\infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln\frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln\frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.