W10 - Examples

Interval of convergence

Find the radius and interval of convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(1) Apply ratio test:

$$\left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| \quad \ggg \quad \frac{n}{n+1}|x-3|$$

Therefore $R = 1$ and thus:

$$|x-3| < 1 \quad \Longrightarrow \quad \text{converges}$$

$$|x-3| > 1 \quad \Longrightarrow \quad \text{diverges}$$

Preliminary interval: $x \in (2, 4)$.

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(2) Check endpoints:

Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \quad \ggg \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\ggg \quad \text{converges by AST}$$

Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \quad \ggg \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\ggg \quad \text{diverges as } p\text{-series}$$

Final interval of convergence: $x \in [2, 4)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(1) Apply ratio test:

$$\left|\frac{\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n x^n}{\sqrt{n+1}}}\right| \quad \gg\gg \quad \frac{|(-3)(-3)^n|}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{|(-3)^n|}|x|$$

$$\gg\gg \quad \frac{3\sqrt{n+1}}{\sqrt{n+2}}|x|$$

Therefore:

$$|x| < \frac{1}{3} \quad \implies \quad \text{converges}$$

$$|x| > \frac{1}{3} \quad \implies \quad \text{diverges}$$

Preliminary interval: $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

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(2) Check endpoints:

Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(-\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{diverges by LCT with } b_n = 1/\sqrt{n}$$

Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(+\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$\gg\gg \quad \text{converges by AST}$$

Final interval of convergence: $x \in \big(-1/3, 1/3\big]$

Interval of convergence - further examples

Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Ratio of terms:

$$\frac{n+1}{3^{n+2}} \cdot \frac{3^{n+1}}{n}|x+2| \quad \gg\gg \quad \frac{n+1}{3n}|x+2|$$

Therefore $R = 3$ and the preliminary interval is $x \in (-5, 1)$.

Check endpoints: $\displaystyle\sum \frac{n(-3)^n}{3^{n+1}}$ diverges and $\displaystyle\sum \frac{n(3)^n}{3^{n+1}}$ also diverges.

Final interval is $(-5, 1)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Ratio of terms:

$$\frac{\frac{1}{n+1}}{\frac{1}{n}}|4x+1| \quad \gg\!\gg \quad \frac{n}{n+1}|4x+1|$$

Therefore:

$$\begin{array}{lclcl} |4x+1| < 1 & \iff & |x+1/4| < 1/4 & \implies & \text{converges} \\ |4x+1| > 1 & \iff & |x+1/4| > 1/4 & \implies & \text{diverges} \end{array}$$

Preliminary interval: $x \in (0, 1/2)$

Check endpoints: $\displaystyle\sum \frac{(4 \cdot \frac{-1}{2} + 1)^n}{n}$ converges but $\displaystyle\sum \frac{1}{n}$ diverges.

Final interval of convergence: $[-1/2, 0)$

Manipulating geometric series algebra

Find power series that represent the following functions:

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1+x^2}$ (c) $\dfrac{x^3}{x+2}$ (d) $\dfrac{3x}{2-5x}$

Solution

(a) $\dfrac{1}{1+x}$

(1) Rewrite in format $\frac{1}{1-u}$.

Introduce double negative:

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Choose $u = -x$.

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(2) Plug $u = -x$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

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(3) Plug in $u = -x$:

$$\gg\!\gg \qquad 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

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(4) Simplify:

$$\gg\!\gg \qquad 1 - x + x^2 - x^3 + \cdots$$

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(5) Final answer:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(b) $\dfrac{1}{1+x^2}$

(1) Rewrite in format $\frac{1}{1-u}$.

Rewrite:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Choose $u = -x^2$.

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(2) Plug $u = -x^2$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = -x^2$:

$$\gg\!\gg \quad 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$\gg\!\gg \quad 1 - x^2 + x^4 - x^6 + \cdots$$

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(3) Final answer:

$$\frac{1}{1+x} = 1 - x^2 + x^4 - x^6 + \cdots$$

(c) $\dfrac{x^3}{x+2}$

(1) Rewrite in format $Ax^3 \cdot \frac{1}{1-u}$.

Rewrite:

$$\frac{x^3}{x+2} \qquad \gg\!\gg \qquad x^3 \cdot \frac{1}{2+x} \qquad \gg\!\gg \qquad x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$\gg\!\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1+\frac{x}{2}} \qquad \gg\!\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

Choose $u = -\frac{x}{2}$. Here $Ax^3 = \frac{1}{2}x^3$.

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(2) Plug $u = -x^2$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = -\frac{x}{2}$:

$$\gg\gg \qquad 1 + (-\tfrac{x}{2}) + (-\tfrac{x}{2})^2 + (-\tfrac{x}{2})^3 + \cdots$$

$$\gg\gg \qquad 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

Obtain:

$$\frac{1}{1 - \left(-\frac{x}{2}\right)} = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

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(3) Multiply by $\frac{1}{2}x^3$.

Distribute:

$$\frac{1}{2}x^3 \cdot \frac{1}{1 - \left(-\frac{x}{2}\right)} \qquad \gg\gg \qquad \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

Final answer:

$$\frac{x^3}{x+2} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

(d) $\dfrac{3x}{2-5x}$

(1) Rewrite in format $Ax \cdot \frac{1}{1-u}$.

Rewrite:

$$\frac{3x}{2-5x} \qquad \gg\gg \qquad 3x \cdot \frac{1}{2-5x}$$

$$\gg\gg \qquad 3x \cdot \frac{1}{2\left(1 - \frac{5x}{2}\right)} \qquad \gg\gg \qquad \frac{3}{2}x \cdot \frac{1}{1 - \frac{5x}{2}}$$

Choose $u = \frac{5x}{2}$. Here $Ax = \frac{3}{2}x$.

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(2) Plug $u = \frac{5x}{2}$ into geometric series.

Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

Plug in $u = \frac{5x}{2}$:

$$\gg\gg \qquad 1 + (\tfrac{5x}{2}) + (\tfrac{5x}{2})^2 + (\tfrac{5x}{2})^3 + \cdots$$

$$\gg\gg \qquad 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

Obtain:

$$\frac{1}{1-\frac{5x}{2}} = 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

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(3) Multiply by $\frac{3}{2}x$.

Distribute:

$$\frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}} \qquad \gg\gg \qquad \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Final answer:

$$\frac{3x}{2-5x} = \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Recognizing and manipulating geometric series Part I

(a) Evaluate $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.
(Hint: consider the series of $\ln(1-x)$.)

(b) Find a series approximation for $\ln(2/3)$.

Solution

(a)

(1) We know the series of $\frac{-1}{1-x}$:

$$\frac{-1}{1-x} = -(1+x+x^2+\cdots) = -1-x-x^2-\cdots$$

Notice that $\int \frac{-1}{1-x}\,dx = \ln(1-x)+C$; this is the desired function when $C=0$.

Integrate the series term-by-term:

$$\int \frac{-1}{1-x}\,dx = \int -1-x-x^2-\cdots\,dx$$

$$\gg\gg \quad \ln(1-x) = D - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

Solve for D using $\ln(1-0)=0$, so $0 = D-0-0-\cdots$ and thus $D=0$. So:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad = \quad \sum_{n=1}^{\infty} -\frac{x^n}{n!}$$

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(2) Notice the formula:

The series formula $\sum_{n=1}^{\infty} -\frac{x^n}{n!}$ looks similar to the formula $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

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(3) Choose $x=-1$ to recreate the desired series:

We obtain equality by setting $x = -1$ because $-(-1)^n = (-1)^{n+1} = (-1)^{n-1}$.

Final answer is $\ln(1 - -1) = \ln 2$.

(b)

Find a series approximation for $\ln(2/3)$:

(1) Observe that $\ln(2/3) = \ln(1 - 1/3)$.

Therefore we can use the series $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

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(2) Plug $x = 1/3$ into the series for $\ln(1-x)$.

Plug in and simplify:

$$\begin{aligned}\ln(2/3) = \ln(1 - 1/3) &= -1/3 - \frac{(1/3)^2}{2} - \frac{(1/3)^3}{3} - \cdots \\ &= -\frac{1}{3} - \frac{1}{3^2 \cdot 2} - \frac{1}{3^3 \cdot 3} - \cdots\end{aligned}$$

Recognizing and manipulating geometric series Part II

(a) Find a series representing $\tan^{-1}(x)$ using differentiation.

(b) Find a series representing $\displaystyle\int \frac{dx}{1+x^4}$.

Solution

(a)

(1) Notice that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

Obtain the series for $\frac{1}{1+x^2}$.

Let $u = -x^2$:

$$\frac{1}{1+x^2} \qquad \gg\gg \qquad \frac{1}{1-u} = 1 + u + u^2 + \cdots$$

$$\gg\gg \qquad 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

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(2) Integrate the series for $\frac{1}{1+x^2}$ by terms.

Set up the strategy. We know:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

and:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Integrate the series term-by-term:

$$\gg\!\gg \quad \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots \, dx$$

$$\gg\!\gg \quad D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Conclude:

$$\tan^{-1}(x) + C = D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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(3) Solve for $D - C$ by testing at $\tan^{-1}(0) = 0$.

Plug in:

$$\tan^{-1}(0) \;=\; D - C + 0 + \cdots + 0$$

$$\gg\!\gg \quad D - C = 0$$

Final answer: $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(b)

(1) Find a series representing the integrand.

Integrand is $\dfrac{1}{1+x^4}$.

Rewrite integrand in format of geometric series sum:

$$\frac{1}{1+x^4} \qquad \gg\!\gg \qquad \frac{1}{1-(-x^4)} \qquad \gg\!\gg \qquad \frac{1}{1-u}, \quad u = -x^4$$

Write the series:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$$

$$\gg\!\gg \quad 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

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(2) Integrate the series by terms:

$$\int 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \, dx \qquad \gg\!\gg \qquad C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots$$