W11 - Examples

Maclaurin series of e to the x

What is the Maclaurin series of $f(x) = e^x$?

Solution

Because $\frac{d}{dx}e^x = e^x$, we find that $f^{(n)}(x) = e^x$ for all n.

So $f^{(n)}(0) = e^0 = 1$ for all n. Therefore $a_n = \frac{1}{n!}$ for all n by the Derivative-Coefficient identity.

Thus:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Maclaurin series of cos x

Find the Maclaurin series representation of $\cos x$.

Solution

Use the Derivative-Coefficient Identity to solve for the coefficients:

$$a_n \quad = \quad \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/24$
5	$-\sin x$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By studying the generating pattern of the coefficients, we find for the series:

$$\cos x \quad = \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Maclaurin series from other Maclaurin series

(a) Find the Maclaurin series of $\sin x$ using the Maclaurin series of $\cos x$.

(b) Find the Maclaurin series of $f(x) = x^2 e^{-5x}$ using the Maclaurin series of e^x.

(c) Using (b), find the *value* of $f^{(22)}(0)$.

Solution

(a)

Remember that $\frac{d}{dx}\cos x = -\sin x$

Differentiate $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Differentiate term-by-term:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \gg\gg \quad 0 - 2\frac{x^1}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$

$$= \quad -\frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

Take negative because $\sin x = -\frac{d}{dx}\cos x$:

$$\gg\gg \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Final answer is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(b)

(1)

Recall the series $e^u = 1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$

Compute the series for e^{-5x}.

Set $u = -5x$:

$$1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

$$\gg\gg \quad 1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots$$

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(2) Compute the product.

Product of series:

$$x^2e^{-5x} \quad \gg\gg \quad x^2\left(1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots\right)$$

$$\gg\gg \quad x^2 - 5x^3 + \frac{25}{2}x^4 - \frac{125}{3!}x^5 + \cdots$$

$$\gg\gg \quad \sum_{n=0}^{\infty}(-1)^n\frac{5^n x^{n+2}}{n!}$$

(c)

(1)

⚠ Derivatives at $x = 0$ are calculable from series coefficients.

Suppose we know the series $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

Then $f^{(n)}(0) = n! \cdot a_n$.

It may be easier to compute a_n for a given $f(x)$ than to compute the derivative *functions* $f^{(n)}(x)$ and then evaluate them.

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(2) Compute a_{22}.

Write the series such that it reveals the coefficients:

$$\sum_{n=0}^{\infty}(-1)^n \frac{5^n x^{n+2}}{n!} \qquad \gg\gg \qquad \sum_{n=0}^{\infty}\left((-1)^n \frac{5^n}{n!}\right)x^{n+2}$$

$$\implies \qquad a_{n+2} = (-1)^n \frac{5^n}{n!}$$

> **Coefficient with a_{n+2} corresponds to the term with x^{n+2}, *not necessarily* the $(n+2)^{\text{th}}$ term (e.g. if the first term is x^2 as here).**

Compute a_{22}:

$$a_{22} = (-1)^{20} \frac{5^{20}}{20!} \qquad \gg\gg \qquad 5^{20} \frac{1}{20!}$$

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(3) Compute $f^{(22)}(0)$.

Use Derivative-Coefficient Identity:

$$f^{(22)}(0) \quad = \quad 22! \cdot a_{22}$$

$$\gg\gg \quad 5^{20} \cdot \frac{22!}{20!} \quad \gg\gg \quad 5^{20} \cdot 22 \cdot 21$$

Computing a Taylor series

Find the first five terms of the Taylor series of $f(x) = \sqrt{x+1}$ centered at $c = 3$.

Solution

A Taylor series is just a Maclaurin series that isn't centered at $c = 0$.

The general format looks like this:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

The coefficients satisfy $a_n = \frac{f^{(n)}(c)}{n!}$. (Notice the c.)

We find the coefficients by computing the derivatives and evaluating at $x = 3$:

$$\begin{aligned} f(x) &= (x+1)^{1/2}, & f(3) &= 2 \\ f'(x) &= \frac{1}{2}(x+1)^{-1/2}, & f'(3) &= \frac{1}{4} \\ f''(x) &= -\frac{1}{4}(x+1)^{-3/2}, & f''(3) &= -\frac{1}{32} \\ f'''(x) &= \frac{3}{8}(x+1)^{-5/2}, & f'''(3) &= \frac{3}{256} \\ f^{(4)}(x) &= -\frac{15}{16}(x+1)^{-7/2}, & f^{(4)}(3) &= -\frac{15}{2048} \end{aligned}$$

By dividing by $n!$ we can write out the first terms of the series:

$$f(x) \;=\; \sqrt{x+1}$$

$$=\; 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \frac{5}{16,384}(x-3)^4 + \cdots$$

Taylor polynomial approximations

Let $f(x) = \sin x$ and let $T_n(x)$ be the Taylor polynomials expanded around $c = 0$.

By considering the alternating series error bound, find the first n for which $T_n(0.02)$ must have error less than 10^{-6}.

Solution

(1) Write the Maclaurin series of $\sin x$ because we are expanding around $c = 0$.

Alternating sign, odd function:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

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(2)

⚠ **Notice this series is alternating, so AST error bound formula applies.**

AST error bound formula is:

$$|E_n| \leq a_{n+1}$$

Here the series is $S = a_0 - a_1 + a_2 - a_3 + \cdots$ and $E_n = S - S_n$ is the error.

✎ **Notice that $x = 0.02$ is part of the terms a_i in this formula.**

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(3) Implement error bound to set up equation for n.

Find n such that $a_{n+1} \leq 10^{-6}$, and therefore by the AST error bound formula:

$$|E_n| \leq a_{n+1} \leq 10^{-6}$$

Plug in $x = 0.02$.

From the series of $\sin x$ we obtain for a_{2n+1}:

$$a_{2n+1} = \frac{0.02^{2n+1}}{(2n+1)!}$$

We seek the first time it happens that $a_{2n+1} \leq 10^{-6}$.

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(4) Solve for the first time $a_{2n+1} \leq 10^{-6}$.

Equations to solve:

$$\frac{0.02^{2n+1}}{(2n+1)!} \leq 10^{-6} \qquad \text{but:} \quad \frac{0.02^{2(n-1)+1}}{(2(n-1)+1)!} \not\leq 10^{-6}$$

Method: list the values:

$$\frac{0.02^1}{1!} = 0.02, \qquad \frac{0.02^3}{3!} \approx 1.33 \times 10^{-6},$$

$$\frac{0.02^5}{5!} \approx 2.67 \times 10^{-11}, \qquad \ldots$$

The first time a_{2n+1} is below 10^{-6} happens when $2n+1 = 5$.

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(5) Interpret result and state the answer.

When $2n+1 = 5$, the term $\dfrac{x^{2n+1}}{(2n+1)!}$ at $x = 0.02$ is less than 10^{-6}.

Therefore the sum of prior terms is accurate to an error of less than 10^{-6}.

The sum of prior terms equals $T_4(0.02)$.

Since $T_4(x) = T_3(x)$ because there is no x^4 term, the same sum is $T_3(0.02)$.

The final answer is $n = 3$.

It would be wrong to infer at the beginning that the answer is 5, or to solve $2n+1 = 5$ to get $n = 2$.

Taylor polynomials to approximate a definite integral

Approximate $\displaystyle\int_0^{0.3} e^{-x^2}\,dx$ using a Taylor polynomial with an error no greater than 10^{-5}.

Solution

(1) Write the series of the integrand.

Plug $u = -x^2$ into the series of e^u:

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots$$

$$\gg\gg \qquad e^{-x^2} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

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(2) Compute definite integral by terms.

Antiderivative by terms:

$$\int 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\,dx$$

$$\gg\gg \quad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Plug in bounds for definite integral:

$$\int_0^{0.3} e^{-x^2}\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \Big|_0^{0.3}$$

$$\gg\gg \qquad 0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} - \frac{0.3^7}{7!} + \cdots$$

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(3) Notice AST, apply error formula.

Compute some terms:

$$\frac{0.3^3}{3!} \approx 0.0045, \qquad \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}, \qquad \frac{0.3^7}{7!} \approx 4.34 \times 10^{-8}$$

So we can guarantee an error less than 4.34×10^{-5} by summing the first terms through $\frac{0.3^5}{5!}$.

Final answer is $0.3 - \dfrac{0.3^3}{3!} + \dfrac{0.3^5}{5!} \approx 0.291243$.