1. Suppose a code must be created using 5 letters and 3 numerals. (There are 26 letters and 10 numerals.) Symbol order matters for codes.

(a) How many unique codes are possible if the 5 letters must come first and the 3 numerals last, and symbols *cannot* be repeated?

26P5 for the letters, 10P3 for the numbers. Multiply them.

26P5 * 10P3

= 7893600 * 720

= 5683392000

(b) How many unique codes are possible if any ordering pattern of 5 letters and 3 numerals is permitted, and symbols *cannot* be repeated?

Pick the 5 spots for the letters first, then repeat part (a)

8C5 * 26P5 * 10P3

= 56 * 7893600 * 720

= 318269952000

(c) How many unique codes are possible if any ordering pattern of 5 letters and 3 numerals is permitted, and symbols *can* be repeated?

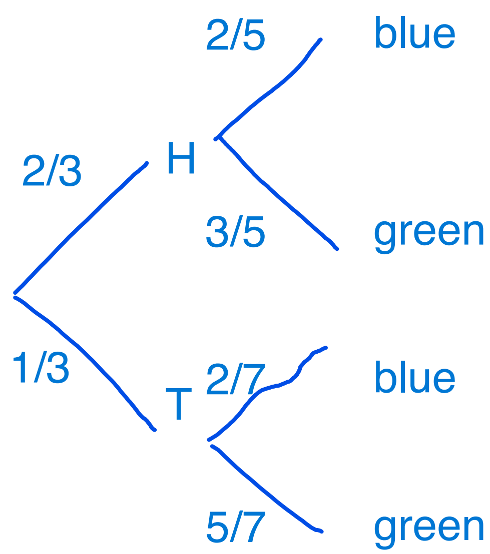
Pick the 5 spots for the letters first, then 26^5 ways for the letters and 10^3 ways for the numbers

56×26^5×10^3

= 665357056000

2. Box A has 2 blue and 3 green marbles. Box B has 2 blue and 5 green marbles. You toss a biased coin with P(head) = 2/3. If it lands heads, choose Box A; if tails, choose Box B. Then, you draw a marble from the chosen box.

(a) What is the probability that a blue marble will be drawn?



P(blue)

= P(blue & H) + P(blue & T)

= 4/15 + 2/21

= 38/105

= 0.3619

(b) If a blue marble is drawn, what is the probability that Box A was chosen?

P(H | blue) = P(blue & H) / P(blue)

= (4/15) / (38/105) = 14/19 = 0.7368

3. Suppose a call center receives calls throughout the day at an average rate of 6 calls per hour according to a Poisson process.

(a) What is the probability that there are no calls between 9:00 and 9:30 AM?

30 mins = 0.5 hours => a = 3

$X \sim \text{Poisson}(3)$

$P(X=0) = \exp(-3)*(3^0)/0! = \exp(-3) = 0.0498$

(b) What is the probability that the 14th call of the day comes in 20 minutes or less after the 11th call?

20 mins = 1/3 hours => a = 2

$X \sim \text{Poisson}(2)$

$P(X >= 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \exp(-2) - \exp(-2)*2 - \exp(-2)*4/2$

$= 1 - 5*\exp(-2)$

$= 0.3233$

4. The automatic opening device of a cargo parachute has been designed to open when the parachute is 200m above ground. The opening altitude has a Normal distribution with a mean value of 200m and standard deviation of 40m. Equipment damage will occur if the parachute opens at an altitude of less than 100m.

(a) What is the probability that there is equipment damage to the payload of a parachute?

$X \sim N(200, 40)$

$P(X<100) = P(Z < -100/40) = P(Z < -2.5)$

$= 1 - 0.99379 = 0.0062$

(b) If 100 cargo parachutes are dropped, find the probability that more than one parachute will open at 100m or less, leading to damaged cargo.

$Y \sim Bin(100, 0.0062)$

$P(Y > 1) = 1 - P(Y=0) - P(Y=1)$

= 1 - 0.0062^0*0.9938^100 - 100 * 0.0062*0.9938^99

$= 0.1281$

5. In a digital transmission, the probability that a bit has high, moderate, and low distortion is 0.01, 0.03, and 0.96 respectively. Suppose that 2 bits are transmitted and that the distortion of each bit is independent of the other. Let X and Y denote the number of bits with high and moderate distortion, respectively, out of the 2 transmitted bits.

a) Find $P_{X,Y}(x,y)$, the joint PMF.

$y \backslash x$	0	1	2	P_Y
0	0.9216	0.0192	0.0001	0.9409
1	0.0576	0.0006	0	0.0582
2	0.0009	0	0	0.0009
P_X	0.9801	0.0198	0.0001	

$0.96^2 = 0.9216$

$0.96 \cdot 0.03 \cdot (2 \text{ ways}) = 0.0576$

$0.96 \cdot 0.01 \cdot (2 \text{ ways}) = 0.0192$

(b) Find $\hat{x}_M(y)$, the *exact* minimal mean-square error estimate of X given $Y =$ ~~x~~ 1

$P[Y=1] = 0.0582 = P_Y(1)$

$E[X \mid Y=1] = 0 \cdot P_{X|Y}(0,1) + 1 \cdot P_{X|Y}(1,1) + 2 \cdot P_{X|Y}(2,1)$

$= 0 + \frac{0.0006}{0.0582} + 2 \cdot \frac{0}{0.0582}$

$= 0.01031$

(c) Find $\hat{X}_L(Y)$, the *linear* minimal mean-square error estimator function for X given Y.

$$\hat{X}_L(Y) = \frac{Cov[X,Y]}{Var[Y]}(Y - \mu_Y) + \mu_X$$

Y \ X	0	1	2	P_Y
0	0.9216	0.0192	0.0001	0.9409
1	0.0576	0.0006	0	0.0582
2	0.0009	0	0	0.0009
P_X	0.9801	0.0198	0.0001	

$$\mu_X = 0.0198 + 2 \cdot 0.0001 = 0.02$$

$$\mu_Y = 0.0198 + 2 \cdot 0.0009 = 0.06$$

$$E[XY] = 0.0006$$

$$E[Y^2] = 1 \cdot 0.0582 + 4 \cdot 0.0009 = 0.0618$$

$$Var[Y] = 0.0618 - 0.06^2 = 0.0582$$

$$Cov[X,Y] = 0.0006 - (0.02)(0.06) = -0.0006$$

$$\frac{Cov[X,Y]}{Var[Y]} = \frac{-0.0006}{0.0582} = -0.01031$$

$$\hat{X}_L(Y) = -0.01031 \cdot (Y - 0.06) + 0.02$$

OR

$$= -0.01031 \cdot Y + 0.02062$$

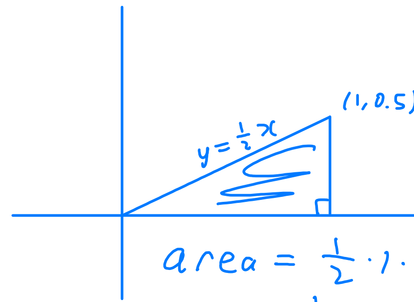
6. Darts are thrown at a triangular dart board and are guaranteed to land somewhere on the board. Let *(X, Y)* be the coordinates (as random variables) of the landing point, and $0 \le 2y \le x \le 1$. So the three vertices of the dart board are (0, 0), (1, 0) and (1, 0.5).

(a) Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le 2y \le x \le 1 \\ 0 & otherwise \end{cases}$$

What is the value of c?

$c = \frac{1}{Area} = 4$



area $= \frac{1}{2} \cdot 1 \cdot \frac{1}{2}$

$= \frac{1}{4}$

(b) Are X and Y independent? (Justify your answer.)

$f_X = \int_{y=0}^{\frac{1}{2}x} 4\,dy = 4y\Big|_0^{\frac{1}{2}x} = 2x$ for $0 \le x \le 1$

$f_Y = \int_{x=2y}^{1} 4\,dx = 4x\Big|_{2y}^{1} = 4 - 8y$ for $0 \le y \le \frac{1}{2}$

$f_X \cdot f_Y \ne 4$ not independent

Alternative: $f_{X,Y}$ = constant

but shape is not X-Y symmetric

info about X gives info about Y

e.g.

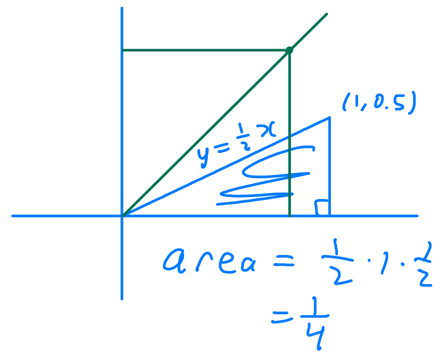
$P[X \approx 0.75 \mid Y \approx 0.4]$

$\ne P[X \approx 0.75]$

(c) It happens that your chance of winning a prize improves as the landing point approaches the upper-right corner (x = 1, y = 0.5) of the triangle. Find the PDF of $W = max(X, Y)$.

(The max function picks the larger of the two values x and y, and assigns it to w.)

Method 1



$$F_W(w) = P[X \le w \text{ AND } Y \le w]$$

$$= \int_{x=0}^{w} \int_{y=0}^{\frac{1}{2}x} 4 \, dy \, dx$$

$$= \int_{x=0}^{w} 2x \, dx = x^2 \Big|_0^w = w^2 \quad \text{for } 0 \le w \le 1$$

$$f_W(w) = \begin{cases} 2w & w \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Method 2

$Max(X, Y) = X$ because $2Y \le X$

Thus $f_W = f_X = 2x$ etc.

7. Let the continuous RV X represent the weight of a newborn baby in pounds. We can assume that $P(X < 0) = 0$ and $P(X < 20) = 1$. Let $F(x)$ be the CDF and $f(x)$ the PDF of X. Which of the following conclusions are <u>NOT</u> justifiable with the information given? (Circle all such statements.)

(a) $F(x) = 0$ for all $x \leq 0$.

C and F are NOT justifiable.

(b) $F(x) = 1$ for all $x \geq 20$.

Everything else is good.

(c) The area under the graph of $F(x)$ between $x = 0$ and $x = 20$ is 1.

(d) $F(x)$ is a non-decreasing function between $x = 0$ and $x = 20$.

(e) No portion of the graph of $f(x)$ can lie below the x-axis.

(f) The height of $f(x)$ cannot exceed 1 between $x = 0$ and $x = 20$.

8. A company sells cars at an average price of \$24,550 with standard deviation \$158 but an unknown distribution. Determine which of the following probabilities can be approximately computed by applying the Central Limit Theorem. (Circle all correct items.)

B and C are correct, A is not.

(a) The probability that a randomly selected car will sell for more than \$25,000.

(b) The probability that a randomly selected shipment of 100 cars from this company sells for less than a total of \$2,500,000.

(c) The probability that the average car price in a randomly selected shipment of 100 cars will be less than \$25,000.

9. You roll a fair die 200 times and count the total number of times you get a six. What is the probability that you will get 35 or more sixes?

(Hint: You may consider using the Central Limit Theorem. You should use the De Moivre-Laplace Continuity Correction Formula / Refined Approximation Method if appropriate.)

$K \sim Bin(200, 1/6)$

$E(K) = np = 100/3$
$V(K) = np(1-p) = 500/18 = 250/9$

$K \sim N$ with mean = 100/3 and variance = 250/9 using CLT

$P(K >= 35)$

$= P(K > 34.5)$ with the approximation method

$= P(Z > 0.22)$

$= 1 - P(Z < 0.22)$

$= 1 - 0.5871$

$= 0.4129$

10. You roll a fair die. Let N denote the number rolled ($N = 1, 2, 3, 4, 5, 6$). Now toss a fair coin N times. Let S be the number of heads observed.

What is $E(S)$?

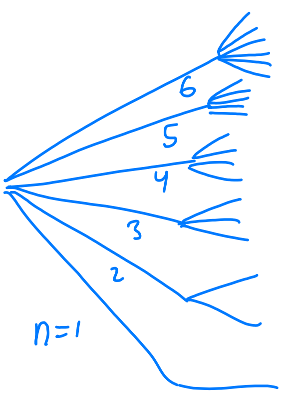
Method 1

$$E[S] = E[E[S|N]]$$

$$E[S|N=n] = \begin{cases} 3 & n=6 \\ 2.5 & n=5 \\ 2 & n=4 \\ 1.5 & n=3 \\ 1 & n=2 \\ 0.5 & n=1 \end{cases} = \frac{n}{2}$$

$$E[S|N] = \frac{N}{2} \qquad E[S] = \frac{1}{2}E[N] = \frac{1}{2}\cdot\frac{7}{2} = \frac{7}{4}$$

Method 2



$$E[S] = \sum_{i=0}^{6} s\,P[S=i]$$

$$P_S(0) = \frac{1}{6}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\right)$$

$$P_S(1) = \frac{1}{6}\left(\frac{1}{2}+2\frac{1}{4}+3\frac{1}{8}+4\frac{1}{16}+5\frac{1}{32}+6\frac{1}{64}\right)$$

$$P_S(2) = \frac{1}{6}\left(0+\frac{1}{4}+3\frac{1}{8}+6\frac{1}{16}+\binom{5}{2}\frac{1}{32}+\binom{6}{2}\frac{1}{64}\right)$$

$$P_S(3) = \frac{1}{6}\left(0+0+\frac{1}{8}+4\frac{1}{16}+\binom{5}{3}\frac{1}{32}+\binom{6}{3}\frac{1}{64}\right)$$

$$P_S(4) = \frac{1}{6}\left(0+0+0+5\frac{1}{16}+\binom{6}{4}\frac{1}{64}\right)$$

$$P_S(5) = \frac{1}{6}\left(0+0+0+0+6\frac{1}{64}\right)$$

$$P_S(6) = \frac{1}{6}\left(0+0+0+0+\frac{1}{64}\right)$$

$$E[S] = 0\cdot P_S(0) + 1\cdot P_S(1) + \ldots + 6\cdot P_S(6)$$
$$= \frac{7}{4} \text{ presumably...}$$

11. A signal of strength X is emitted by one of two beacons, H_0 or H_1. The sample space of X is [0,1]. The <u>CDF</u> of X, conditioned on the emitting beacon, is:

$$F_{X|H0}(x) = 1 - (1-x)^3 , \qquad 0 \leq x \leq 1$$

$$F_{X|H1}(x) = x^3, \qquad 0 \leq x \leq 1$$

The prior probability of beacon H_0 being activated is 0.20. Assuming that we want to minimize the total probability of error, choose a hypothesis test type, and state the decision rule clearly indicating the decision threshold and the two acceptance sets.

f(XIH0) = 3(1-x)^2, 0 <= x <= 1
f(XIH1) = 3x^2, 0 <= x <= 1

P(H0) = 0.2, P(H1) = 0.8

MAP test:

x belongs to A0 if f(XIH0)*P(H0) >= f(XIH1)*P(H1)

or, if 3(1-x)^2*0.2 >= 3*x^2*0.8

or, if (1-x)^2 >= 4*x^2

or, if 1 - x >= 2x

or, if x <= 1/3

A0/ accept H0 if x <= 1/3
A1/ accept H1 otherwise

Use this page if you need extra space for a problem. Make a note on that page; otherwise work on this page will not be graded.