APMA - Probability Formula Sheet

Distribution Data

Discrete

In all formulas, $q = 1 - p$.

Bernoulli:

$$X \sim \text{Ber}(p): \qquad P_X(k) = \begin{cases} p & k = 1 \\ q & k = 0 \\ 0 & \text{else} \end{cases}$$

$$E[X] = p, \qquad \text{Var}[X] = pq$$

Binomial:

$$X \sim \text{Bin}(n, p): \qquad E[X] = np, \qquad \text{Var}[X] = npq$$

Geometric:

$$X \sim \text{Geo}(p): \qquad P_X(k) = \begin{cases} pq^{k-1} & k = 1, 2, 3, \ldots \\ 0 & \text{else} \end{cases}$$

$$E[X] = \frac{1}{p}, \qquad \text{Var}[X] = \frac{q}{p^2}$$

Pascal:

$$X \sim \text{Pascal}(\ell, p): \qquad P_X(k) = \begin{cases} \binom{k-1}{\ell-1} p^\ell q^{k-\ell} & k = \ell, \ell+1, \ell+2, \ldots \\ 0 & \text{else} \end{cases}$$

$$E[X] = \frac{\ell}{p}, \qquad \text{Var}[X] = \frac{\ell q}{p^2}$$

Poisson:

$$X \sim \text{Pois}(\lambda): \qquad P_X(k) = \begin{cases} \dfrac{\lambda^k e^{-\lambda}}{k!} & k = 0, 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

$$E[X] = \lambda, \qquad \text{Var}[X] = \lambda$$

Continuous

Exponential:

$$X \sim \text{Exp}(\lambda): \qquad f_X(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$E[X] = \frac{1}{\lambda}, \qquad \text{Var}[X] = \frac{1}{\lambda^2}$$

Erlang:

$$X \sim \text{Erlang}(\ell, \lambda): \qquad f_X(t) = \begin{cases} \dfrac{\lambda^\ell}{(\ell-1)!} t^{\ell-1} e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$E[X] = \frac{\ell}{\lambda}, \qquad \text{Var}[X] = \frac{\ell}{\lambda^2}$$

Normal:

$$X \sim \mathcal{N}(\mu, \sigma^2), \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Uniform:

$$X \sim \text{Unif}[a, b], \qquad f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$E[X] = \frac{a+b}{2}, \qquad \text{Var}[X] = \frac{(b-a)^2}{12}$$

Tail Estimation

Markov inequality: Assume $X \geq 0$ for all outcomes, and $c > 0$ is any constant. Then:

$$P[X \geq c] \;\leq\; \frac{E[X]}{c}$$

Chebyshev inequality: Assume $c > 0$ is any constant. Then:

$$P\Big[\,|X - \mu_X| \geq c\,\Big] \;\leq\; \frac{\text{Var[X]}}{c^2}$$

Linear MSE Estimator

The optimal linear MSE estimator function of X given Y is:

$$\hat{X}_L(Y) \;=\; a^* Y + b^*$$

$$a^* = \rho_{X,Y} \frac{\sigma_X}{\sigma_Y} = \frac{\text{Cov}[X, Y]}{\text{Var}[X]}, \qquad b^* = \mu_X - a^* \mu_Y$$

The expected error of this estimator is:

$$e_L^* \;=\; E\left[\left(X - \hat{X}_L(Y)\right)^2\right] \;=\; \sigma_X^2 \left(1 - \rho_{X,Y}^2\right)$$