W08 Notes

Functions on two random variables

01 Theory

PMF of any function of two variables

Suppose $W = g(X, Y)$ and X, Y are discrete RVs.

The PMF of W:

$$P_W(w) = \sum_{\substack{(x,y) \text{ s.t.} \\ g(x,y)=w}} P_{X,Y}(x, y)$$

CDF of continuous function of two variables

Suppose $W = g(X, Y)$ and X, Y are continuous RVs, and g is a continuous function.

The CDF of W:

$$F_W(w) \;=\; P[W \leq w] \;=\; \iint\limits_{g(x,y)\leq w} f_{X,Y}(x, y)\, dxdy$$

One can then compute the PDF of W by differentiation:

$$f_W(w) \;=\; \frac{d}{dw} F_W(w)$$

02 Illustration

Example - PDF of a quotient

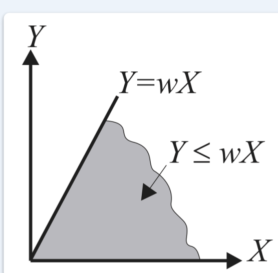
Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x, y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)} & x,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $W = Y/X$.

Solution

(1) Find the CDF using logic:



Convert to event form:

$$F_W(w) \;=\; P[Y/X \leq w]$$

$$\gg\gg \quad P[Y \leq wX]$$

Integrate over this region:

$$\begin{aligned}
P[Y \leq wX] \quad &= \quad \int_0^\infty \int_0^{wx} f_{X,Y}(x,y)\,dy\,dx \\
&\gg\gg \quad \int_0^\infty \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx \\
&\gg\gg \quad \int_0^\infty \lambda e^{-\lambda x}\left(-e^{-\mu w x} + 1\right)dx \\
&\gg\gg \quad 1 - \frac{\lambda}{\lambda + \mu w}
\end{aligned}$$

---

(2) Differentiate to find PDF:

Compute $\frac{d}{dw}F_W(w)$:

$$f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda+\mu w)^2} & w \geq 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Exercise - PMF of XY^2 from chart

Suppose the joint PMF of X and Y is given by this chart:

$Y\downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = XY^2$.

(a) Find the PMF $P_W(w)$.

(b) Find the expectation $E[W]$.

Example - Max and Min from joint PDF

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} \frac{3}{2}(x^2+y^2) & x,\,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find the PDFs:

(a) $W = \text{Max}(X,Y)$

(b) $W = \text{Min}(X,Y)$

Solution

(a)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \quad = \quad P\big[\text{Max}(X,Y) \le w\big]$$

$$\ggg \quad P\big[X \le w \text{ and } Y \le w\big]$$

Integrate PDF over the region, assuming $w \in [0,1]$:

$$\int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x,y)\,dx\,dy$$

$$\ggg \quad \int_0^w \int_0^w \tfrac{3}{2}(x^2+y^2)\,dx\,dy \quad \ggg \quad w^4$$

---

(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \; = \; P\big[\text{Min}(X,Y) \le w\big]$$

$$\ggg \quad 1 - P\big[\text{Min}(X,Y) > w\big]$$

$$\ggg \quad 1 - P[X > w \text{ and } Y > w]$$

Integrate PDF over the region:

$$P[X > w \text{ and } Y > w] \quad \ggg \quad \int_w^1 \int_w^1 \tfrac{3}{2}(x^2+y^2)\,dx\,dy$$

$$\ggg \quad w^4 - w^3 - w + 1$$

Therefore:

$$F_W(w) \; = \; w + w^3 - w^4$$

---

(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 1 + 3w^2 - 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$