W10 Notes

Review: True/False

TRUE or FALSE:

(a) Suppose $\mathrm{Cov}[X,Y] = 0.05$. It is possible that the correlation coefficient $\rho_{X,Y} = 0.05$.

(b) Suppose $\mathrm{Cov}[X,Y] = 0.05$. It is possible that X and Y have a strong linear relationship.

(c) Suppose $\mathrm{Cov}[X,Y] = 0$. It is possible that X and Y are *not* independent.

(d) Suppose X and Y are *not* independent. It is possible that $\mathrm{Cov}[X,Y]$ is equal to 0.

(e) Suppose X and Y *are* independent. $\mathrm{Cov}[X,Y]$ must be equal to 0.

Review: Conditional probability

Recall some items related to conditional probability.

Conditioning definition:

$$P[- \mid A] \quad = \quad \frac{P[- \overset{\text{"AND"}}{\cap} A]}{P[A]}$$

Multiplication rule:

$$P[AB] \quad = \quad P[B \mid A]\, P[A]$$

Division into Cases / Total Probability:

$$P[B] \quad = \quad P[B \mid A_1]\, P[A_1] + \cdots + P[B \mid A_n]\, P[A_n]$$

Conditional distribution

01 Theory

Conditional distribution - fixed event

Suppose X is a random variable, and suppose $A \subset \mathbb{R}$. The distribution of X **conditioned on A** describes the probabilities of values of X given knowledge that $X \in A$.

Discrete case:

$$P_{X|A}(k) \;=\; \begin{cases} \dfrac{1}{P[A]} P_X(k) & k \in A \\ 0 & k \notin A \end{cases}$$

$\rightarrow P[X = k \mid A]$

Continuous case:

$$f_{X|A}(x) \;=\; \begin{cases} \dfrac{1}{P[A]} f_X(x) & x \in A \\ \\ 0 & x \notin A \end{cases}$$

There is also a conditional CDF, of which this conditional PDF is the derivative:

$$F_{X|A}(x) \;=\; P[X \leq x \mid A], \qquad f_{X|A}(x) \;=\; \frac{d}{dx} F_{X|A}(x)$$

The Law of Total Probability has versions for distributions:

$$\begin{aligned} P_X(k) \quad &= \quad P_{X|A_1}(k)\, P[A_1] + \cdots + P_{X|A_n}(k)\, P[A_n] \\ f_X(x) \quad &= \quad f_{X|A_1}(x)\, P[A_1] + \cdots + f_{X|A_n}(x)\, P[A_n] \end{aligned}$$

Conditional distribution - variable event

Suppose X and Y are any two random variables. The **distribution of X conditioned on Y** describes the probabilities of values of X in terms of y, given knowledge that $Y = y$.

Discrete case:

$$\begin{aligned} P_{X|Y}(k|\ell) \;&=\; P[X = k \mid Y = \ell] \\ &=\; \frac{P_{X,Y}(k,\ell)}{P_Y(\ell)} \qquad \text{(assuming } P_Y(\ell) \neq 0\text{)} \end{aligned}$$

i.e. $P[A]$ w/ $A =$ "$Y = \ell$"

Continuous case:

$$f_{X|Y}(x|y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{(assuming } f_Y(y) \neq 0\text{)}$$

Remember: $P_{X,Y}(k,\ell)$ is the probability that "$X = k$ and $Y = \ell$."

Sometimes it is useful to have the formulas rewritten like this:

$$\begin{aligned} P_{X,Y}(k,\ell) \quad &= \quad P_{X|Y}(k|\ell)\, P_Y(\ell) \\ f_{X,Y}(x,y) \quad &= \quad f_{X|Y}(x|y)\, f_Y(y) \end{aligned}$$

Like $P[BA] = P[\underset{X=k}{\underset{\shortparallel}{B}}|A]P[\overset{Y=\ell}{\overset{\shortparallel}{A}}]$

Extra - Deriving $f_{X|Y}(x|y)$

The density $f_{X|Y}$ ought to be such that $f_{X|Y}(x|y)\,dx$ gives the probability of $X \in [x, x+dx]$, given knowledge that $Y \in [y, y+dy]$. Calculate this probability:

$$P\Big[x \leq X \leq x + dx \;\Big|\; y \leq Y \leq y + dy\Big]$$

$$\gg\!\gg \quad \frac{P\Big[x \leq X \leq x + dx,\; y \leq Y \leq y + dy\Big]}{P\Big[y \leq Y \leq y + dy\Big]}$$

$$\gg\!\gg \quad \frac{f_{X,Y}(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\gg\!\gg \quad \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

02 Illustration

☰ Example - Conditional PMF, variable event, via joint density

Suppose X and Y have joint PMF given by:

$$P_{X,Y}(k,\ell) = \begin{cases} \dfrac{k+\ell}{21} & k=1,2,3;\ \ell=1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P_{X|Y}(k|\ell)$ and $P_{Y|X}(\ell|k)$.

Solution

$$\sum_{\ell=1,2} \frac{k+\ell}{21} = \frac{k+1}{21} + \frac{k+2}{21} = \frac{2k+3}{21}$$

Marginal PMFs:

$$P_X(k) = \frac{2k+3}{21}, \qquad k=1,2,3$$

$$P_Y(\ell) = \frac{\ell+2}{7}, \qquad \ell=1,2$$

Assuming $\ell = 1$ or 2, for each $k = 1$, 2, 3 we have:

$$P_{X|Y}(k|\ell) = \frac{P_{X,Y}(k,\ell)}{P_Y(\ell)} \gg\gg \frac{k+\ell}{3\ell+6}$$

Assuming $k = 1$, 2, or 3, for each $\ell = 1, 2$ we have:

$$P_{Y|X}(\ell|k) = \frac{P_{Y,X}(\ell,k)}{P_X(k)} \gg\gg \frac{k+\ell}{2k+3}$$

Conditional expectation

03 Theory

Expectation conditioned by a fixed event

Suppose X is a random variable and $A \subset \mathbb{R}$. The **expectation of X conditioned on A** describes the typical value of X given the hypothesis that $X \in A$ is known.

Discrete case:

$$E[\,X \mid A\,] = \sum_k k\, P_{X|A}(k)$$

$$E[\,g(X) \mid A\,] = \sum_k g(k)\, P_{X|A}(k)$$

Continuous case:

$$E[\,X \mid A\,] = \int_{-\infty}^{+\infty} x\, f_{X|A}(x)\, dx$$

$$E[\,g(X) \mid A\,] = \int_{-\infty}^{+\infty} g(x)\, f_{X|A}(x)\, dx$$

Conditional variance:

$$\mathrm{Var}[\,X \mid A\,] \quad = \quad E\Big[(X - \overset{E[x|A]}{\mu_{X|A}})^2 \mid A\Big] \quad = \quad E[X^2 \mid A] - \mu_{X|A}^2$$

Division into Cases / Total Probability applied to expectation:

$$E[X] \quad = \quad E[\,X \mid A_1\,]\,P[A_1] + \cdots + E[\,X \mid A_n\,]\,P[A_n]$$

$f_x = f_{x|A_1} \cdot P[A_1] + \dots + f_{x|A_n} \cdot P[A_n]$

Linearity of conditional expectation:

$$E[\,aX_1 + bX_2 + c \mid Y = y\,] = a\,E[\,X_1 \mid Y = y\,] + b\,E[\,X_2 \mid Y = y\,] + c$$

Extra - Proof: Division of Expectation into Cases

We prove the discrete case only.

1. Expectation formula:

$$E[X] \quad = \quad \sum_k k\,P_X(k)$$

2. Division into Cases for the PMF:

$$P_X(k) \quad = \quad \sum_{i=1}^{n} P_{X|A_i}(k)\,P[A_i]$$

3. Substitute in the formula for $E[X]$:

$$\sum_k k\,P_X(k) \quad \ggg \quad \sum_k k \sum_{i=1}^{n} P_{X|A_i}(k)\,P[A_i]$$

$$\ggg \quad \sum_{i=1}^{n} P[A_i] \sum_k k\,P_{X|A_i}(k)$$

$$\ggg \quad \sum_{i=1}^{n} P[A_i]\,E[\,X \mid A_i\,]$$

Expectation conditioned by a variable event

Suppose X and Y are any two random variables. The **expectation of X conditioned on $Y = y$** describes the typical of value of X in terms of y, given the hypothesis that $Y = y$ is known.

Discrete case:

$$E[\,X \mid Y = y\,] \quad = \quad \sum_k k\,\overbrace{P_{X|Y}(k|y)}^{P[x=x \mid Y=y]} \quad (k \text{ over all poss. vals.})$$

$$E[\,g(X,Y) \mid Y = y\,] \quad = \quad \sum_k g(k,y)\,P_{X|Y}(k|y)$$

Continuous case:

$$E[\,X \mid Y = y\,] \quad = \quad \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx$$

$$E[\,g(X,Y) \mid Y = y\,] \quad = \quad \int_{-\infty}^{+\infty} g(x,y)\, f_{X|Y}(x|y)\, dx$$

05 Illustration

☰ Example - Conditional PMF, fixed event, expectation

Suppose X measures the lengths of some items and has the following PMF:

$$P_X(k) \quad = \quad \begin{cases} 0.15 & k = 1,2,3,4 \\ 0.1 & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Let L be the event that $X \geq 5$. $L = "X \geq 5"$

(a) Find the conditional PMF of X given that L is known.

(b) Find the conditional expected value and variance of X given L.

Solution

(a)

Conditional PMF formula with $\overset{k}{\cancel{x}} \in L$ plugged in:

$$P_{X|L}(\overset{k}{\cancel{x}}) \quad = \quad \begin{cases} \dfrac{P_X(\overset{k}{\cancel{x}})}{P[L]} & \overset{k}{\cancel{x}} = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Compute $P[L]$ by adding cases:

$$P[L] \;=\; \sum_{k=5}^{8} P_X(k) \quad \gg\!\gg \quad 0.4$$

Divide nonzero PMF entries by 0.1:

$$P_{X|L}(k) \quad = \quad \begin{cases} 0.25 & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

(b)

Find $E[\,X \mid L\,]$:

$$E[\,X \mid L\,] \;=\; \sum_{k=5}^{8} k\, P_{X|L}(k)$$

$$\gg\!\gg \quad 5\cdot(0.25) + 6\cdot(0.25) + 7\cdot(0.25) + 8\cdot(0.25)$$

$$\gg\!\gg \quad 6.5\,\text{min}$$

Find $E[\,X^2 \mid L\,]$:

$$E[\,X^2 \mid L\,] \;=\; \sum_{k=5}^{8} k^2\,P_{X|L}(k)$$

$$\gg\gg \quad 5^2\cdot(0.25)+6^2\cdot(0.25)+7^2\cdot(0.25)+8^2\cdot(0.25)$$

$$\gg\gg \quad 43.5\,\mathrm{min}^2$$

Find $\mathrm{Var}[\,X \mid L\,]$ using “short form” with conditioning:

$$\mathrm{Var}[\,X \mid L\,] \quad = \quad E[\,X^2 \mid L\,] - E[\,X \mid L\,]^2 \quad \gg\gg \quad 1.25\,\mathrm{min}^2$$

04 Theory - extra

Expectation conditioned by a random variable

Suppose X and Y are any two random variables. The **expectation of X conditioned on Y** is a random variable giving the typical value of X on the assumption that Y has value determined by an outcome of the experiment.

$$E[\,X \mid Y\,] \quad = \quad g(Y) \quad \text{where } \; g(y) = E[\,X \mid Y = y\,]$$

In other words, start by defining a function $g(y)$:

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ y &\mapsto E[\,X \mid Y = y\,] \end{aligned}$$

Now $E[\,X \mid Y\,]$ is defined as the composite random variable $g(Y)$.

Considered as a random variable, $E[\,X \mid Y\,]$ takes an outcome $s \in S$, computes $Y(s)$, sets $y = Y(s)$, then returns the expectation of X conditioned on $Y = y$.

Notice that X is *not* evaluated at s, only Y is.

Because the value of $E[\,X \mid Y\,]$ depends only on $Y(s)$, and not on any additional information about s, it is common to *represent* a conditional expectation $E[\,X \mid Y\,]$ using only the function g.

Iterated Expectation

$$E[\;E[X \mid Y]\;] = E[X]$$

Proof of Iterated Expectation, discrete case

$$\begin{aligned} E[\,E[\,X \mid Y\,]\,] &= \sum_{\ell} E[X \mid Y = \ell]\,P_Y(\ell) \\ &= \sum_{\ell}\sum_{k} k\,P_{X|Y}(k|\ell)\,P_Y(\ell) \\ &= \sum_{k} k \sum_{\ell} P_{X,Y}(k,\ell) \\ &= \sum_{k} k\,P_X(k) = E[X] \end{aligned}$$

05 Illustration - extra

Example - Conditional expectations from joint density

Suppose X and Y are random variables with joint density given by:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{y} e^{-x/y} e^{-y} & x, y \in (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Find $E[\,X \mid Y = y\,]$. Use this to compute $E[X]$.

Solution

(1) Derive the marginal density $f_Y(y)$:

$$\begin{aligned} f_Y(y) \quad &\gg\!\!\gg \quad \int_0^{+\infty} \tfrac{1}{y} e^{-x/y} e^{-y}\,dx \\ \gg\!\!\gg \quad -e^{-x/y} e^{-y}\Big|_{x=0}^{\infty} \quad &\gg\!\!\gg \quad e^{-y} \end{aligned}$$

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(2) Use $f_Y(y)$ to compute $f_{X|Y}(x|y)$:

$$\begin{aligned} f_{X|Y}(x|y) \quad &\gg\!\!\gg \quad \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ \gg\!\!\gg \quad \tfrac{1}{y} e^{-x/y} e^{-y} \cdot (e^{-y})^{-1} \quad &\gg\!\!\gg \quad \tfrac{1}{y} e^{-x/y} \end{aligned}$$

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(3) Use $f_{X|Y}(x|y)$ to calculate expectation conditioned on the variable event:

$$\begin{aligned} E[X \mid Y = y] \quad &\gg\!\!\gg \quad \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\,dx \\ \gg\!\!\gg \quad \int_0^{\infty} \tfrac{x}{y} e^{-x/y}\,dx \quad &\gg\!\!\gg \quad y \end{aligned}$$

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(4) Apply Iterated Expectation:

Set $g(y) = y$. By Iterated Expectation, we know that $E[X] = E[g(Y)]$. Therefore:

$$E[X] = E[g(Y)] \quad = \quad \int_{-\infty}^{+\infty} g(y)\, f_Y(y)\, dy$$

$$\gg\!\gg \quad \int_0^{+\infty} y\, e^{-y}\, dy \quad \gg\!\gg \quad 1$$

Notice that $g(Y) = Y$, so $E[\,X \mid Y\,] = Y$, and Iterated Expectation says that $E[X] = E[Y]$.

Example - Flip coin, choose RV

Suppose $X \sim \text{Ber}(1/3)$ and $Y \sim \text{Ber}(1/4)$ represent two biased coins, giving 1 for heads and 0 for tails.

Here is the experiment:

1. Flip a fair coin.
2. If heads, flip the X coin; if tails, flip the Y coin.
3. Record the outcome as Z.

What is $E[Z]$?

Solution

Let $G \sim \text{Ber}(1/2)$ describe the fair coin. Then:

$$E[Z] \;=\; E[\,E[Z \mid G\,]\,]$$

$$\gg\!\gg \quad E[\,Z \mid G = 0\,]\, P_G(0) + E[\,Z \mid G = 1\,]\, P_G(1)$$

$$\gg\!\gg \quad E[Y]\, P_G(0) + E[X]\, P_G(1)$$

$$\gg\!\gg \quad \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \quad \gg\!\gg \quad \frac{7}{24}$$

Example - Sum of random number of RVs

Let N denote the number of customers that enter a store on a given day.

Let X_i denote the amount spent by the i^{th} customer. $i = 1, 2, \ldots, N$

Assume that $E[N] = 50$ and $E[X_i] = \$8$ for each i.

What is the expected total spend of all customers in a day?

Solution

A formula for the total spend is $X = \sum_{i=1}^{N} X_i$.

By Iterated Expectation, we know $E[X] = E[\,E[\,X \mid N\,]\,]$.

Now compute $E[X \mid N]$ as a function of N:

$$E[X \mid N = n] \quad \gg\gg \quad E\left[\left(\sum_{i=1}^{N} X_i\right) \mid N = n\right]$$

$$\gg\gg \quad E\left[\left(\sum_{i=1}^{n} X_i\right) \mid N = n\right]$$

$$\gg\gg \quad \sum_{i=1}^{n} E[X_i \mid N = n]$$

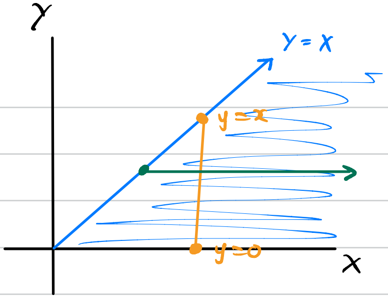
$$\gg\gg \quad \sum_{i=1}^{n} E[X_i] \quad \gg\gg \quad 8n$$

Therefore $g(n) = 8n$ and $g(N) = 8N$ and $E[\,X \mid N\,] = 8N$.

Then by Iterated Expectation, $E[X] = E[8N] = 8E[N] = \$400$.

07-Combo Q 4

$$f_{X,Y} = \begin{cases} 2e^{-(x+2y)} & x, y > 0 \\ 0 & \text{else} \end{cases}$$



Want $P[X > Y]$

$$\int_{x=0}^{\infty} \int_{y=0}^{x} 2e^{-(x+2y)} \, dy \, dx$$

-OR-

$$= \int_{y=0}^{\infty} \int_{x=y}^{\infty} 2e^{-(x+2y)} \, dx \, dy$$

$$\int_{x=y}^{\infty} 2e^{-(x+2y)} \, dx = \lim_{R \to \infty} \int_{y}^{R} 2e^{-(x+2y)} \, dx = \lim_{R \to \infty} -2e^{-(x+2y)} \Big|_{y}^{R}$$

$$= \lim_{R \to \infty} -2e^{-(R+2y)} - -2e^{-3y}$$

$$= 2ye^{-3y}$$

Suppose X and Y are random variables with joint density given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y} e^{-x/y} e^{-y} & x, y \in (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Find $E[\, X \mid Y = y\,]$. Use this to compute $E[X]$.

$$f_Y = \int_{-\infty}^{+\infty} f_{X,Y} \, dx$$

1\. Find $f_Y(y) = \int_{x=0}^{\infty} \frac{1}{y} e^{-x/y} e^{-y} \, dx = e^{-y}$ for $y > 0$

$= 0$ else

2\. Compute $f_{X|Y}(x|y) = \frac{f_{X,Y}}{f_Y} = \begin{cases} \frac{1}{y} e^{-x/y} & x, y > 0 \\ 0 & \text{else} \end{cases}$

3\. $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y} \, dx$

$$= \int_{x=0}^{\infty} \frac{x}{y} e^{-x/y} \, dx = \boxed{y} \quad \rightsquigarrow g(y) = y$$

4\. So: $E[X|Y] = g(Y) = \boxed{Y}$