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APMA 3100 Sec 003 Probability
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What is Probability? (for a mathematician)
Ingredients: <u>Probability Model</u>

- Sample Space $S = \{1, 2, 3, 4, 5, 6\}$
- Events $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \ldots, \{1,2\}, \{1,3\}, \ldots\}$
$= \{A, B, C, \ldots\}$
- Probability Measure $P : \mathcal{F} \to \mathbb{R}$ following rules

"Kolmogorov Axioms":

1) $P[A] \geq 0$ 2) $P[S] = 1$

3) $P[A \cup B] = P[A] + P[B]$ when $A \cap B = \emptyset$
also infinite: $P[A_1 \cup A_2 \cup \ldots] = P[A_1] + P[A_2] + \ldots$ when $A_i \cap A_j = \emptyset$

Example Roll die

$A =$ "even number" $= \{2, 4, 6\}$
$B =$ "≤ 3" $= \{1, 2, 3\}$

$A \cup B = \{1, 2, 3, 4, 6\}$

$(A \cup B)^c = \{5\} = A^c \cap B^c = \{1, 3, 5\} \cap \{4, 5, 6\}$
$= \{5\}$

Q: Roll two dice. What is $|S|$?
A: 36

Sample Space (2 dice):

Red = X
Green = Y

Y \ X =	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)		. . .
4	(1,4)	(2,4)	(3,4)	. . .		
5	⋮					
6				(4,6)		(6,6)

Y \ X =	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)		. . .
4	(1,4)	(2,4)	(3,4)	. . .		
5	⋮					
6				(4,6)		(6,6)

$X+Y=4$

Y \ X =	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)		. . .
4	(1,4)	(2,4)	(3,4)	. . .		
5	⋮					
6				(4,6)		(6,6)

$X+Y \leq 5$

Y \ X =	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)		. . .
4	(1,4)	(2,4)	(3,4)	. . .		
5	⋮					
6				(4,6)		(6,6)

$X \leq 2$

Note: singleton events
e.g. $\{(4,3)\} = E$ contains one outcome

These matter for discrete probability.

Probability: e.g.

$$P[\{(4,3)\}] = \frac{1}{36} = \frac{|E|}{|S|}$$

$$P[X+Y=4] = \frac{3}{36} = \frac{1}{12} = 3\left(\frac{1}{36}\right)$$

$$P[X+Y \leq 5] = 10\left(\frac{1}{36}\right) = \frac{5}{18}$$

$$P[X \leq 2] = 12\left(\frac{1}{36}\right) = \frac{6}{18} = \frac{1}{3}$$

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Commonly Used Facts

- **Negation:** $P[A^c] = 1 - P[A]$

  e.g. $P[\text{"not 2"}] = \frac{5}{6} = 1 - P[\text{"2"}] = 1 - \frac{1}{6} = \frac{5}{6}$

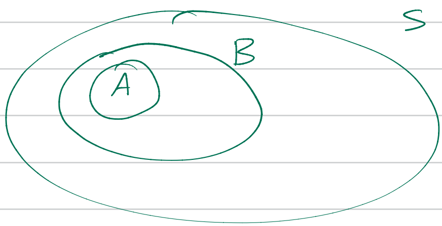
- **Monotonicity:** $A \subset B \Rightarrow P[A] \leq P[B]$

  $P[A \cup B \setminus A] = P[B]$

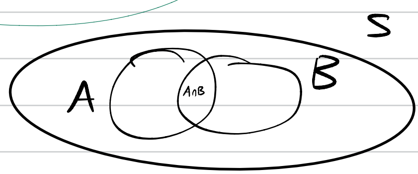
  $\parallel$

  $P[A] + \underbrace{P[B \setminus A]}_{\geq 0}$

  $\Rightarrow P[A] \leq P[B]$



- **Inclusion-Exclusion Rule:**



$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$\rightsquigarrow P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

Thus called "inclusion-exclusion"

Example At UVA, 25% students have iPhone
30% have iPad
60% have neither.

Q1: Find prob. random student has <u>some</u> iProduct.
Q2: Find prob. . . . . . . <u>both</u> iProducts.

Solution:

1. Set up model: Outcomes = {chosen student's products}
   Events = sets of students
   P: all students equally likely

2. Label events:

O = student has iPhone
A = " " iPad

Note: $P[OA] = P[O \cap A]$

Q1: $P[O \cup A] = ?$ Q2: $P[O \cap A] = ?$

3. Interpret Data:

Have: $P[O] = 0.25$
$P[A] = 0.30$

$$P[\text{"neither"}] = 0.60 = P[(O \cup A)^c] = P[O^c \cap A^c]$$

4. Break union with "IE":

Notice: $$P[O \cup A] = \underbrace{P[O]}_{0.25} + \underbrace{P[A]}_{0.30} - P[O \cap A] \quad \text{(IE)}$$

5. Apply negation, solve:

$$\underbrace{P[(O \cup A)^c]}_{= 0.60} = 1 - P[((O \cup A)^c)^c] = 1 - P[O \cup A]$$

$$\boxed{P[O \cup A] = 0.40} \quad Q1$$

Thus: $$P[O \cap A] = -(0.40 - (0.25 + 0.30)) = 0.55 - .40 = \boxed{0.15} \quad Q2$$

Conditional Probability

$$P[B \mid A] = \text{“Prob. of } B \text{ given } A\text{”} = \frac{P[B \cap A]}{P[A]}$$

Also: a new model $P_A[B] = P[B \mid A]$

or $P[- \mid A]$

FACT:

$P[- \mid A]$ is a valid P, satisfies Kolmogorov Axioms

(K2)

A_1, A_2

$A_1 \cap A_2 = \emptyset$

$$P_A[A_1 \cup A_2] \overset{?}{\underset{?}{=}} P_A[A_1] + P_A[A_2]$$

$$\frac{P[(A_1 \cup A_2) \cap A]}{P[A]} \overset{?}{\underset{?}{=}} \frac{P[A_1 \cap A]}{P[A]} + \frac{P[A_2 \cap A]}{P[A]}$$

i.e. $P[A_1 A \cup A_2 A] \overset{?}{=} P[A_1 A] + P[A_2 A]$

✓ true bc $A_1 A \cap A_2 A = \emptyset$

Example:

Draw 2 cards.

$P[$“1^{st} is a 3” and “2^{nd} is a 4”$]$

$T :=$ “1^{st} card is 3.”

$F :=$ “2^{nd} card is 4.”

Q: Find $P[TF]$.

A: $P[F \mid T] = \dfrac{P[TF]}{P[T]} \rightsquigarrow P[TF] = P[T]\,P[F \mid T]$ “multiplication rule”

$$P[T] = \frac{4}{52}$$

$$P[F \mid T] = \frac{4}{51} \rightsquigarrow \boxed{P[TF] = \frac{4}{52} \cdot \frac{4}{51}}$$