D = "Division into Cases"

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Law of Total Probability

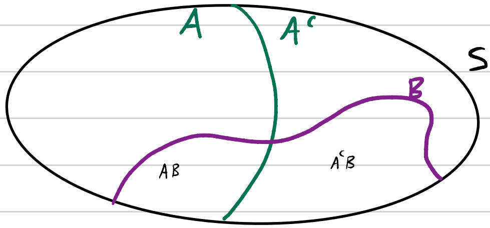
"AB" = "$A \cap B$" = "$A \wedge B$"

$$P[B] = P[A]P[B|A] + P[A^c]P[B|A^c]$$

Recall

$P[AB] = P[A]P[B|A]$

$P[A^cB] = P[A^c]P[B|A^c]$



Example: Two bins, B1 = 5 red, 4 green marbles
B2 = 4 red, 5 green marbles

Experiment: Take one from B1 put in B2. Shake B2.
Take one from B2.

Q: Find P[draw red] = ?

A:

T_R = transfer red $\qquad D_R$ = draw red

T_G = transfer green $\qquad D_G$ = draw green

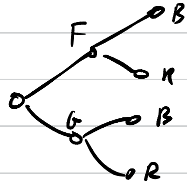
$$\text{Answer} = P[D_R] = P[T_R]P[D_R|T_R] + P[T_G]P[D_R|T_G]$$

$$= \frac{5}{9}\cdot\frac{5}{10} + \frac{4}{9}\cdot\frac{4}{10} = \boxed{\frac{41}{90}}$$

Bayes' Theorem

$$P[B|A] = P[A|B] \cdot \frac{P[B]}{P[A]} = P[B] \frac{P[A|B]}{P[A]}$$

- multistage experiments
- inferences to the middle



Example: Bins like above B1: 5r 4g, B2: 4r 5g
Friend select bin at random, then draw from bin.
Red is drawn. Q: Prob. that he picked bin 1.

A: R, G, B_1, B_2

know: $P[B_1] = P[B_2] = \frac{1}{2}$

know:
$$P[R|B_1] = 5/9$$
$$P[G|B_1] = 4/9$$
$$P[R|B_2] = 4/9$$
$$P[G|B_2] = 5/9$$

want: $P[B_1|R]$

Bayes': $P[B_1|R] = P[R|B_1] \cdot \frac{P[B_1]}{P[R]}$

$$= \frac{5}{9} \cdot \frac{1/2}{?}$$ need $P[R]$

$$P[R] = P[B_1]P[R|B_1] + P[B_2]P[R|B_2]$$
$$= \frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2}\frac{4}{9} = \frac{1}{2}$$

So $P[B_1|R] = \frac{5}{9} \cdot \frac{1/2}{1/2} = \boxed{\frac{5}{9}} > \frac{1}{2} = \frac{4.5}{9}$

Example:

0.5% of pop. has Covid... We have a test.

- True positive on 96% of those who have.
- False pos. on 2% of those who don't.

Bob tests positive. What are odds Bob has Covid?

A: A_N, A_P "actual" $\qquad A_N = A_P^c$

T_N, T_P "test" $\qquad T_N = T_P^c$

know: $P[T_P \mid A_P] = .96$, $\quad P[T_P \mid A_N] = .02$

$P[A_P] = .005 \qquad P[A_N] = .995$

seek: $P[A_P \mid T_P]$

Bayes': ↳ $= P[T_P \mid A_P] \cdot \dfrac{P[A_P]}{P[T_P]}$

$= .96 \dfrac{.005}{P[T_P]}$ ← need

$P[T_P] = P[A_P]\,P[T_P \mid A_P] + P[A_N]\,P[T_P \mid A_N]$

$= (.005)(.96) + (.995)(.02) \approx .025 \approx 2.5\%$

So $\text{Ans} = (.96)\dfrac{.005}{(.005)(.96)+(.995)(.02)} \approx \boxed{19\%}$