

W03 - Examples

Repeated trials

Multinomial: Soft drinks preferred

Folks coming to a party prefer Coke (55%), Pepsi (25%), or Dew (20%). If 20 people order drinks in sequence, what is the probability that exactly 12 have Coke and 5 have Pepsi and 3 have Dew?

Solution

The multinomial coefficient $\binom{20}{12, 5, 3}$ gives the number of ways to assign 20 people into bins according to preferences matching the given numbers, $C = 12$ and $P = 5$ and $D = 3$.

Each such assignment is one sequence of outcomes. All such sequences have probability $(0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3$.

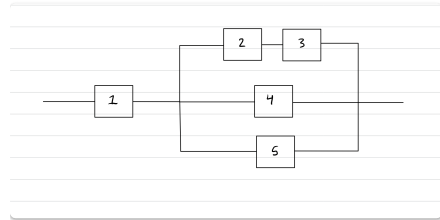
The answer is therefore:

$$\binom{20}{12, 5, 3} \cdot (0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3 \gg \gg \frac{20!}{12! 5! 3!} \cdot (0.55)^{12} \cdot (0.25)^5 \cdot (0.2)^3$$

Reliability

Reliability: Series, parallel, series

Suppose a process has internal components arranged like this:



Write W_i for the event that component i succeeds, and W_i^c for the event that it fails.

The success probabilities for each component are given in the chart:

1	2	3	4	5
92%	89%	95%	86%	91%

Find the probability that the entire system succeeds.

Solution

(1) Conjoin components 2 and 3 in series.

Compute:

$$P[W_2 W_3] \gg \gg P[W_2] \cdot P[W_3] \gg \gg (0.89) \cdot (0.95) = 0.8455$$

Therefore:

$$P[(W_2 W_3)^c] \gg \gg 1 - 0.846 \gg \gg 0.1545$$

(2) Conjoin components (2-3) with 4 and 5 in parallel.

Compute for the complement (failure) first:

$$\begin{aligned} P[(W_2W_3 \cup W_4 \cup W_5)^c] &\ggg P[(W_2W_3)^c] \cdot P[W_4^c] \cdot P[W_5^c] \\ &\ggg (0.1545)(0.14)(0.09) \ggg 0.0019467 \end{aligned}$$

Flip back to success:

$$P[W_2W_3 \cup W_4 \cup W_5] \ggg 1 - 0.0019467 \ggg 0.9980533$$

(3) Conjoin components 1 with (2-3-4-5) in series.

Compute:

$$\begin{aligned} P[W_1(W_2W_3 \cup W_4 \cup W_5)] &\ggg (0.92)(0.9980533) \\ &\ggg 0.918209036 \approx 91.82\% \end{aligned}$$

Discrete random variables

PDF and CDF: Roll 2 dice

Roll two dice colored red and green. Let X_R record the number of dots showing on the red die, X_G the number on the green die, and let S be a random variable giving the total number of dots showing after the roll, namely $S = X_R + X_G$.

- Find the PMFs of X_R and of X_G and of S .
- Find the CDF of S .
- Find $P[S = 8]$.

Solution

(1) Sample space.

Denote outcomes with ordered pairs of numbers (i, j) , where i is the number showing on the red die and j is the number on the green one.

Require that $i, j \in \mathbb{N}$ are integers satisfying $1 \leq i, j \leq 6$.

Events are sets of distinct such pairs.

(2) Create chart of outcomes.

Chart:

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(3) Definitions of X_R , X_G , and S .

We have $X_R(i, j) = i$ and $X_G(i, j) = j$.

Therefore $S(i, j) = i + j$.

(4) Find PMF of X_R .

Use variable n for each possible value of X_R , so $n = 1, 2, \dots, 6$.

Find $P_{X_R}(n)$:

$$P_{X_R}(n) \gg \gg P[X_R = n] \\ \gg \gg \frac{|\text{outcomes with } n \text{ on red}|}{|\text{all outcomes}|} \gg \gg \frac{6}{36} = \frac{1}{6}$$

Therefore $P_{X_R}(n) = 1/6$ for every n .

(5) Find PMF of X_G .

Same as for X_R :

$$P_{X_G}(n) = \frac{1}{6} \quad \text{for all } n$$

(6) Find PMF of S .

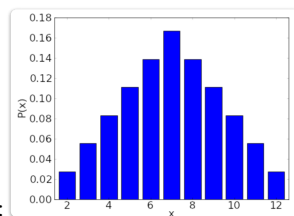
Find $P_S(n)$:

$$P_S(n) \gg \gg P[S = n] \gg \gg \frac{|\text{outcomes with sum } n|}{|\text{all outcomes}|}$$

⚠ Count outcomes along *diagonal lines* in the chart.

Create table of $P_S(n)$:

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k) = P(S = k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



Create bar chart of $P_S(n)$:

Evaluate: $P[S = 8] \gg \gg 5/36$.

(7) Find CDF of S .

CDF definition:

$$F_S(x) = P[S \leq x]$$

Apply definition: add new PMF value at each increment:

$$F_S(n) = \begin{cases} 0 & x < 2 \\ 1/36 & 2 \leq x < 3 \\ 3/36 & 3 \leq x < 4 \\ 6/36 & 4 \leq x < 5 \\ 10/36 & 5 \leq x < 6 \\ 15/36 & 6 \leq x < 7 \\ 21/36 & 7 \leq x < 8 \\ 26/36 & 8 \leq x < 9 \\ 30/36 & 9 \leq x < 10 \\ 33/36 & 10 \leq x < 11 \\ 35/36 & 11 \leq x < 12 \\ 36/36 & 12 \leq x \end{cases}$$

PMF for total heads count; binomial expansion of 1

A fair coin is flipped n times.

Let X be the random variable that counts the total number of heads in each sequence.

The PMF of X is given by:

$$P_X(k) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Since the total probability must add to 1, we know this formula must hold:

$$1 = \sum_{\text{possible } k} P_X(k)$$

$$\gg \gg 1 = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Is this equation really true?

There is another way to view this equation: it is the binomial expansion $(x + y)^n$ where $x = \frac{1}{2}$ and $y = \frac{1}{2}$:

$$\left(\frac{1}{2} + \frac{1}{2}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Life insurance payouts

A life insurance company has two clients, A and B , each with a policy that pays \$100,000 upon death. Consider events D_1 that the older client dies next year, and D_2 that the younger dies next year. Suppose $P[D_1] = 0.10$ and $P[D_2] = 0.05$.

Define a random variable X measuring the total money paid out next year in units of \$1,000. The possible values for X are 0, 100, 200. We calculate:

$$P[X = 0] \gg \gg P[D_1^c]P[D_2^c] = 0.95 \cdot 0.90 = 0.86$$

$$P[X = 100] \gg \gg 0.05 \cdot 0.90 + 0.95 \cdot 0.10 = 0.14$$

$$P[X = 200] \gg \gg 0.05 \cdot 0.10 = 0.005$$