Counting

When all outcomes are equally likely. $P[A] = \frac{|A|}{|S|}$

Permutations:

$$P(n,r) = \frac{n!}{(n-r)!}$$

ways to list r items from n

Combinations:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

ways to group r items from n

$$C(n,r) \cdot r! = P(n,r)$$

Example: # ways to form list of 4 out of 45 permutations.

45	44	43	42
1^{st}	2^{nd}	3^{rd}	4^{th} in the list

$\rightsquigarrow 45 \cdot 44 \cdot 43 \cdot 42$

Observe: $45 \cdot 44 \cdot 43 \cdot 42 = \frac{45!}{41!} = \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot \cancel{41} \cdot \cancel{40} \cdots \cancel{1}}{\cancel{41} \cdot \cancel{40} \cdots \cancel{1}}$

$\hookrightarrow = \boxed{\frac{45!}{(45-4)!}}$

Combinations: # ways to form a team of 4 from the 45 students.

Have $4!$ ways to put team members in a list.

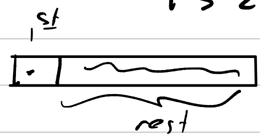
$$C(45,4) \cdot 4! = P(45,4) \rightsquigarrow C(45,4) = \boxed{\frac{45!}{41!} \cdot \frac{1}{4!}}$$

{# teams} × {lists per team} = {# lists}

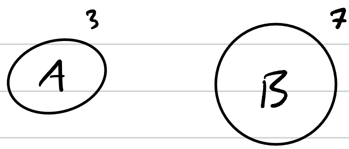
Example: # ways to form team of 5 out of 40 with a chosen leader.

A: $\binom{40}{5}\cdot 5 = \frac{40!}{5!\,35!}\cdot 5 = \frac{40\cdot 39\cdot 38\cdot 37\cdot 36}{4\cdot 3\cdot 2\cdot 1}$

$40\cdot\frac{39\cdot 38\cdot 37\cdot 36}{4\cdot 3\cdot 2\cdot 1}$



Bins:

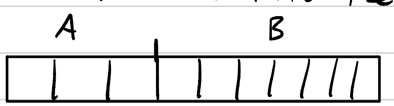


ways to divvy up 10 labeled marbles into these bins

ANS $= \frac{10!}{3!\,7!}$

$= \binom{10}{3} = C(10,3)$

Why?

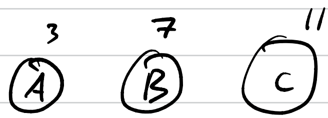


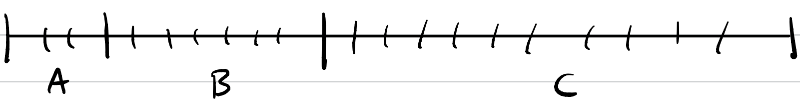
e.g. 2, 8, 9, 3, 1, 5, 4, 7, 6, 10 $\rightsquigarrow$ 10!

÷ by $3!\cdot 7!$

Write $\binom{10}{3,\,7}$

Generalize: e.g. 3 bins:





$\frac{21!}{3!\,7!\,11!} = \binom{21}{3,\,7,\,11}$ = "multinomial coefficient" "binomial" and "multinomial"

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$\vdots$$
$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$= C(n,i)$
$= \frac{n!}{i!(n-i)!}$

$$(x+y)(x+y)(x+y)(x+y)$$

$\rightsquigarrow$ 4 ways to get x^3y
6 ways x^2y^2

Bin A "choose x"

Bin B "choose y"

put factors into bins

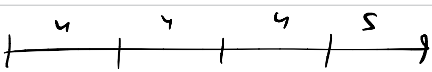
Generalize: $(x+y+z)^n = \sum_{i+j+k=n} \binom{n}{i,j,k} x^i y^j z^k$

Example: Team of 3 out of 40.
Find prob. Cooper is on the team.

A: ANS: $\frac{\#\text{ teams w/ Cooper}}{\#\text{ teams total}} = \frac{\binom{39}{2}}{\binom{40}{3}} = \frac{39!}{2!\,37!} \cdot \frac{3!\,37!}{40!}$
$= \boxed{\frac{3}{40}}$

Example: Board game. 17 ppl. want 4 teams.
Sizes 4, 4, 4, 5. # ways?

A: $\boxed{\frac{17!}{4!\,4!\,4!\,5!}} = \binom{17}{4,4,4,5}$



no I, O, Q

Example: VA License plates: LLL - NNNN
Observe random plate.

(a) Prob. the numbers are strictly increasing.
(b) Prob. at least one repetition of number.

A: Total plates $= 23^3 \cdot 10^4$

(a) $\rightsquigarrow$ no repetition ("strictly")
$\rightsquigarrow$ one increasing ordering per group of 4 distinct from 10

Therefore: $\dfrac{23^3 \cdot \binom{10}{4}}{23^3 \cdot 10^4} = \boxed{\dfrac{21}{1000}}$

(b) "at least one" $\overset{\text{code for}}{\rightsquigarrow}$ not (all distinct)

$E = \geq 1$ repeated

$P[E] = 1 - P[E^c] = 1 - P[\text{all distinct}]$

$$1 - \frac{23^3 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{23^3 \cdot 10^4} = \boxed{0.496}$$

Repeated Trials

Bernoulli Trial: outcome T, F (1, 0)

Ber(p) $\qquad P[T] = p \qquad P[F] = 1-p =: \text{"}q\text{"}$

$0 \le p \le 1$

p = "success prob."

$\left(P[T \cup F] = P[S] = 1 \quad \text{"}P[T] + P[F]\text{"} \right)$

Bernoulli Process: repeat Bernoulli trial n times
– all independent –

Outcomes: "1001101", "0111101", etc.

Indep. $\Rightarrow$ 1st is 1, 2nd is 0 ...

$$P[1001101] = P[1] \cdot P[0] \cdot P[0] \cdots P[1] = pqqppqp = p^4 q^3$$

Example: Flip a biased coin n times, record 1 = heads, 0 = tails
$p = P[1] = P[\text{heads}]$ any single flip.

Random Variables

Binomial random variable: S = number of success in n trials in Bernoulli process

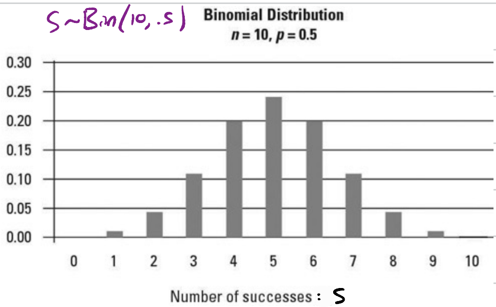
write: $S \sim \text{Bin}(n, p)$ e.g. $n = 7$ above $S(1001101) = 4$

write: $P_S(k) := P[S = k]$ e.g. $k = 1, 2, \dots 7$ when $n = 7$

Example: $S \sim \text{Bin}(7, 0.3)$

$$P_S(4) = P[S=4] = \binom{7}{4} (0.3)^4 (0.7)^3$$

$S \sim \text{Bin}(10, .5)$



$P_S(k)$ "=" $\binom{10}{k} p^k q^{10-k}$

$p = q = 0.5$

sample space $= \Omega$

Def: Random Variable is $X : S \to \mathbb{R}$

Example: Ber. Proc., $X_i = \begin{cases} 1 & i^{th} = 1 \\ 0 & i^{th} = 0 \end{cases}$ (n-times) $\qquad X_i \sim Ber(p)$

PMF: $\quad P_{X_i}(k) = P[X_i = k] = \begin{cases} p & k = 1 \\ q & k = 0 \end{cases}$

Set $\quad S := X_1 + X_2 + \dots + X_n \quad \leadsto \quad S \sim Bin(n, p)$

$S = \#$ success as above

still function: $S(1011) = X_1(1011) + X_2(1011) + X_3(1011) + X_4(1011) = 1 + 0 + 1 + 1 = 3$

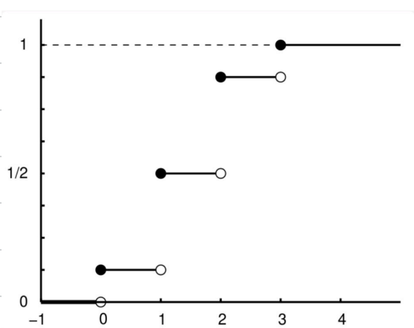
Def: Probability Mass Function, PMF, $\quad P_X(k) := P[X = k]$

Def: Cumulative Distribution Function, CDF:

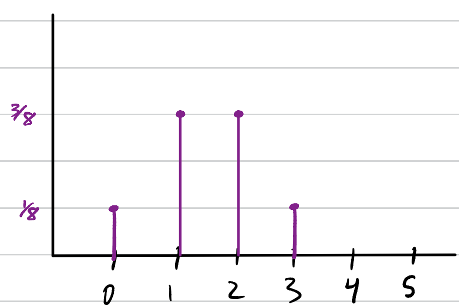
$$F_X(x) = P[X \leq x] \qquad \underset{\text{discrete case:}}{=} \sum_{k_i \leq x} P_X(k_i)$$

$\boxed{\text{CDF is non-decreasing always}}$

Example: Find PMF from this CDF:



Solution:



Example: Roll 2 dice, one red one green.

X_R = red number, X_G = green number

$S = X_R + X_G$

Find: (a) PMF of X_R, X_G, S.

(b) CDF of S (c) $P[S=8]$

ANS: Sample Space $= \{(i,j) \mid i,j = 1,2,\dots,6\}$

$X_R((i,j)) = i$ $X_G((i,j)) = j$

S values:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Solution:

(a) $P_{X_R}(k) = P[X_R = k] = \frac{6}{36} = \frac{1}{6}$ all k

$=$ "constant function" $k=1,2,..6$

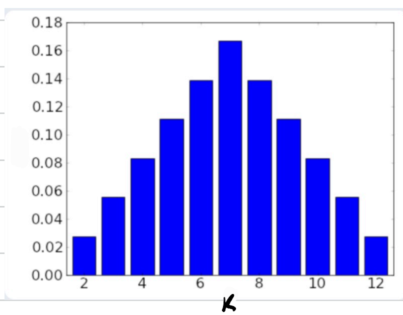
$P_{X_G}(k) = 1/6$ also $k=1,2,..6$

$P_S(k)$: e.g. $P_S(2) = 1/36$

$P_S(6) = 5/36$

k	2	3	4	5	6	7	8	9	10	11	12
$P_S(k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$P_S(k)$



(b) $F_S(x) =$

0	$x < 2$
$1/36$	$2 \le x < 3$
$3/36$	$3 \le x < 4$
$6/36$	$4 \le x < 5$
$10/36$	
$15/36$	$\vdots$
$21/36$	
$26/36$	
$30/36$	
$33/36$	
$35/36$	$11 \le x < 12$
$36/36$	$12 \le x$

(c) $P[S=8] = P_S(8) = \frac{5}{36}$