

W04 - Examples

Bernoulli process

Binomial variable counting ones in repeated die rolls

A standard die is rolled 6 times. Use a binomial variable to find the probability of rolling at least 4 ones.

Solution

(1) Labels.

Let $S \sim \text{Bin}(6, \frac{1}{6})$.

Interpret: S counts the ones appearing over 6 rolls.

We seek $P[4 \leq S]$.

(2) Calculations.

Exclusive events:

$$\begin{aligned}
 P[4 \leq S] &\ggg P_S(4) + P_S(5) + P_S(6) \\
 &\ggg \binom{6}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^2 + \binom{6}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^1 + \binom{6}{6} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^0 \\
 &\ggg \frac{203}{23328} \ggg \approx 0.00870
 \end{aligned}$$

Roll die until

Roll a fair die repeatedly. Find the probabilities that:

- (a) At most 2 threes occur in the first 5 rolls.
- (b) There is no three in the first 4 rolls, using a geometric variable.

Solution

(a)

(1) Labels.

Use $S \sim \text{Bin}(5, 1/6)$ to count the number of threes among the first six rolls.

Seek $P[S \leq 2]$ as the answer.

(2) Calculations.

Divide into exclusive events:

$$\begin{aligned}
 P[S \leq 2] &\ggg P_S(0) + P_S(1) + P_S(2) \\
 &\ggg \binom{5}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 + \binom{5}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4 + \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \\
 &\ggg \frac{625}{648} \ggg \approx 0.965
 \end{aligned}$$

(b)

(1) Labels.

Use $N \sim \text{Geom}(1/6)$ to give the roll number of the first time a three is rolled.

Seek $P[N > 4]$ as the answer.

Sum the PMF formula for $\text{Geom}(1/6)$.

(2) Compute:

$$P[N > 4] \ggg \sum_{k=5}^{\infty} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$

(3)

 **Geometric series formula.**

For any geometric series:

$$a + ar + ar^2 + ar^3 + \dots \ggg \frac{a}{1-r}$$

Apply formula:

$$\sum_{k=5}^{\infty} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right) \ggg \left(\frac{5}{6}\right)^4$$

Final answer is $P[N > 4] = (5/6)^4$.

Cubs winning the World Series

Suppose the Cubs are playing the Yankees for the World Series. The first team to 4 wins in 7 games wins the series. What is the probability that the Cubs win the series?

Assume that for any given game the probability of the Cubs winning is $p = 45\%$ and losing is $q = 55\%$.

Solution

(a) Using a binomial distribution

(1) Label.

Let $X \sim \text{Bin}(7, p)$.

Thus $P_X(4)$ is the probability that the Cubs win exactly 4 games over 7 played.

Seek $P_X(4) + P_X(5) + P_X(6) + P_X(7)$ as the answer.

(2) Calculate.

Use binomial PMF:

$$P_X(k) = \binom{7}{k} p^k q^{7-k}$$

(3) Insert data:

$$\begin{aligned} & P_X(4) + \cdots + P_X(7) \\ \ggg & \binom{7}{4} p^4 q^3 + \binom{7}{5} p^5 q^2 + \binom{7}{6} p^6 q^1 + \binom{7}{7} p^7 q^0 \end{aligned}$$

(4) Compute:

$$\begin{aligned} \ggg & \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} p^4 q^3 + \frac{7 \cdot 6}{2} p^5 q^2 + \frac{7}{1} p^6 q^1 + 1 \cdot p^7 q^0 \\ \ggg & p^4 (35q^3 + 21p^1 q^2 + 7p^2 q + p^3) \end{aligned}$$

Convert $q \gg (1 - p)$:

$$\begin{aligned} \ggg & p^4 (35(1 - p)^3 + 21p(1 - p)^2 + 7p^2(1 - p) + p^3) \\ \ggg & 35p^4 - 84p^5 + 70p^6 - 20p^7 \ggg \approx 0.39 \end{aligned}$$

(b) Using a Pascal distribution

(1) Label.

Let $Y \sim \text{Pasc}(4, p)$.

Thus $P_Y(k)$ is the probability that the Cubs win their 4th game on game number k .

Seek $P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$ as the answer.

(2) Calculate.

Use Pascal PMF:

$$P_Y(k) = \binom{k-1}{3} p^4 q^{k-4}$$

(3) Insert data:

$$\begin{aligned} & P_Y(4) + \dots + P_Y(7) \\ \ggg & \binom{3}{3} p^4 q^0 + \binom{4}{3} p^4 q^1 + \binom{5}{3} p^4 q^2 + \binom{6}{3} p^4 q^3 \end{aligned}$$

(4) Compute:

$$\begin{aligned} \ggg & 1 \cdot p^4 + \frac{4}{1} \cdot p^4 q^1 + \frac{5 \cdot 4}{2} p^4 q^2 + \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} p^4 q^3 \\ \ggg & p^4 (1 + 4q + 10q^2 + 20q^3) \end{aligned}$$

Convert $q \gg (1 - p)$:

$$\begin{aligned} \ggg & p^4 (1 + 4(1 - p) + 10(1 - p)^2 + 20(1 - p)^3) \\ \ggg & 35p^4 - 84p^5 + 70p^6 - 20p^7 \ggg \approx 0.39 \end{aligned}$$

Notice: The algebra seems very different, right up to the end!

Expectation and variance

Gambling game - tokens in bins

Consider a game like this: a coin is flipped; if H then draw a token from Bin 1, if T then from Bin 2.

- Bin 1 contents: 1 token \$1,000, and 9 tokens \$1
- Bin 2 contents: 5 tokens \$50, and 5 tokens \$1

It costs \$50 to enter the game. Should you play it? (A lot of times?) How much would you pay to play?

[Solution](#)

Expected value - rolling dice

Let X be a random variable counting the number of dots given by rolling a single die.

Then:

$$E[X] \ggg 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} \ggg \frac{7}{2}$$

Let S be an RV that counts the dots on a roll of *two* dice.

The PMF of S :

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k) = P(S = k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then:

$$E[S] \ggg 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots + 12 \cdot \frac{1}{36} \ggg 7$$

Notice that $\frac{7}{2} + \frac{7}{2} = 7$.

In general, $E[X + Y] = E[X] + E[Y]$.

Let X be a green die and Y a red die.

From the earlier calculation, $E[X] = \frac{7}{2}$ and $E[Y] = \frac{7}{2}$.

Since $S = X + Y$, we derive $E[S] = 7$ by simple addition!

Expectation from PMF of related variable

Let X have distribution given by this PMF:

x	1	2	3	4	5
$p_X(x)$	$1/7$	$1/14$	$3/14$	$2/7$	$2/7$

Find $E[|X - 2|]$.

Solution

(1) Compute the PMF of $|X - 2|$.

PMF arranged by possible value:

$$\begin{array}{lll}
P[|X-2|=0] & \ggg & P[X=2] = \frac{1}{14} \\
P[|X-2|=1] & \ggg & P[X=1] + P[X=3] = \frac{1}{7} + \frac{3}{14} = \frac{5}{14} \\
P[|X-2|=2] & \ggg & P[X=4] = \frac{2}{7} \\
P[|X-2|=3] & \ggg & P[X=5] = \frac{2}{7} \\
\vdots & & \vdots \\
P[|X-2|=k] & \ggg & 0 \quad \text{for } k \neq 0, 1, 2, 3.
\end{array}$$

(2) Calculate the expectation.

Using formula for discrete PMF:

$$E[|X-2|] = 0 \cdot \frac{1}{14} + 1 \cdot \frac{5}{14} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{2}{7} = \frac{25}{14}.$$

Variance for composite using PMF and simpler formula

Suppose X has this PMF:

$k :$	1	2	3
$P_X(k) :$	1/7	2/7	4/7

Find $\text{Var}[\frac{1}{1+X}]$ using the formula $\text{Var}[Y] = E[Y^2] - E[Y]^2$ with $Y = \frac{1}{1+X}$.

(Hint: you should find $E[Y] = \frac{13}{42}$ and $E[Y^2] = \frac{13}{126}$ along the way.)

Poisson process

Poisson calculation

Suppose $X \sim \text{Pois}(10)$. Find $P[X \leq 13 \mid X \geq 7]$. (Leave the answer in exact form.)

Solution

(1) Conditioning definition:

$$P[X \leq 13 \mid X \geq 7] \ggg \frac{P[7 \leq X \leq 13]}{1 - P[X < 7]}$$

(2) Expand numerator:

$$P[7 \leq X \leq 13] \ggg e^{-10} \frac{10^7}{7!} + e^{-10} \frac{10^8}{8!} + \dots + e^{-10} \frac{10^{13}}{13!}$$

(3) Simplify:

$$\ggg e^{-10} \frac{10^7}{7!} \left(1 + \frac{10}{8} + \frac{10^2}{8 \cdot 9} + \dots + \frac{10^6}{8 \cdot 9 \dots 13} \right)$$

(4) Compute for denominator:

$$P[X < 7] \ggg e^{-10} \frac{10^0}{0!} + e^{-10} \frac{10^1}{1!} + \dots + e^{-10} \frac{10^6}{6!}$$

Arrivals at a post office

Client arrivals at a post office are modelled well using a Poisson variable.

Each potential client has a very low and independent chance of coming to the post office, but there are many thousands of potential clients, so the arrivals at the office actually come in moderate number.

Suppose the average rate is 5 clients per hour.

- (a) Find the probability that nobody comes in the first 10 minutes of opening. (The cashier is considering being late by 10 minutes to run an errand on the way to work.)
- (b) Find the probability that 5 clients come in the first hour. (I.e. the average is achieved.)
- (c) Find the probability that 9 clients come in the first two hours.

Solution

(a)

(1) Convert rate for desired window.

Expect 5/6 clients every 10 minutes.

Let $X \sim \text{Pois}(5/6)$.

Seek $P_X(0)$ as the answer.

(2) Compute.

Formula:

$$P_X(k) = e^{-5/6} \frac{(5/6)^k}{k!}$$

Insert data and compute:

$$P_X(0) \ggg e^{-5/6} \ggg \approx 0.435$$

(b)

Rate is already correct.

Let $X \sim \text{Pois}(5)$.

Compute the answer:

$$P_X(5) = e^{-5} \frac{5^5}{5!} \ggg \approx 0.175$$

(c)

Convert rate for desired window.

Expect 10 clients every 2 hours.

Let $X \sim \text{Pois}(10)$.

Compute the answer:

$$P_X(9) \ggg e^{-10} \frac{10^9}{9!} \ggg \approx 0.125$$

Notice that 0.125 is smaller than 0.175.